Eötvös Loránd University
Faculty of Informatics
Departement of Numerical Analysis

Computer Science Bsc

Basic Mathematics - Functions

University material

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Introduction

Some notations:

- The set of real numbers: $\mathbb{R}$;
- The set of natural numbers: $\mathbb{N} := \{0, 1, 2, \ldots\}$;
- The set of positive integers: $\mathbb{N}^+ := \{1, 2, 3, \ldots\}$;
- The set of integers: $\mathbb{Z} := \mathbb{N} \cup \{-x \in \mathbb{R} :\ x \in \mathbb{N}\}$;
- The set of rational numbers: $\mathbb{Q} := \left\{\dfrac{p}{q} \in \mathbb{R} \ \mid \ p, q \in \mathbb{Z},\ q \neq 0\right\}$;
- The set of complex numbers: $\mathbb{C} := \{z = x + iy \in \mathbb{C} \ \Big| \ x, y \in \mathbb{R}\}$

  (i is the imaginary unit);
- The points of the plane: $\mathbb{R}^2 := \{(x,\, y) \ \mid \ x, y \in \mathbb{R}\}$;
- The points of the space: $\mathbb{R}^3 := \{(x,\, y,\, z) \ \mid \ x, y, z \in \mathbb{R}\}$.

1. Functions, Operations with functions

1.1. Theoretic review

To define the notion of a function, first we need the *Cartesian product* of two sets, and the subsets of this Cartesian product: the so called *relations*.

Ordered pairs

Definition: Let x, y be two "objects" or "elements". The set

$$(x; y) := \{\{x\}; \{x; y\}\}$$

is called an *ordered pair*, where x is the *first component* and y is the *second component*. It can be proved, that two ordered pairs $(x; y)$ and $(a; b)$ are equal to each other if and only if $x = a$ and $y = b$. In formula:

$$(x; y) = (a; b) \iff x = a \ \wedge \ y = b.$$

Cartesian product of sets

Definition: Let $\emptyset \neq A, B$ two arbitrary, nonempty sets. In this case the set:

$$A \times B := \{(x; y) \ \Big| \ x \in A \ \wedge \ y \in B\}$$

is called the *Cartesian product* of the sets A and B. So the elements of $A \times B$ are ordered pairs, whose first component is from A, and the second component comes from B.

Remarks:

1. If $A = B$, then $A \times A$ is denoted by A^2. For example the geometric plane is the set:

   $$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}.$$

   We can also introduce the Cartesian product of 3 sets, as the set of ordered triplets, with elements from the given three sets, in the given order. In this manner

   $$\mathbb{R}^3 := \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

   denotes the usual set of the 3−dimensional points of the space.

2. **For example:**

   $$(2; 1); (-2; 0); (0; 0); (0; -\pi) \in \mathbb{R}^2$$

   are points of the plane and

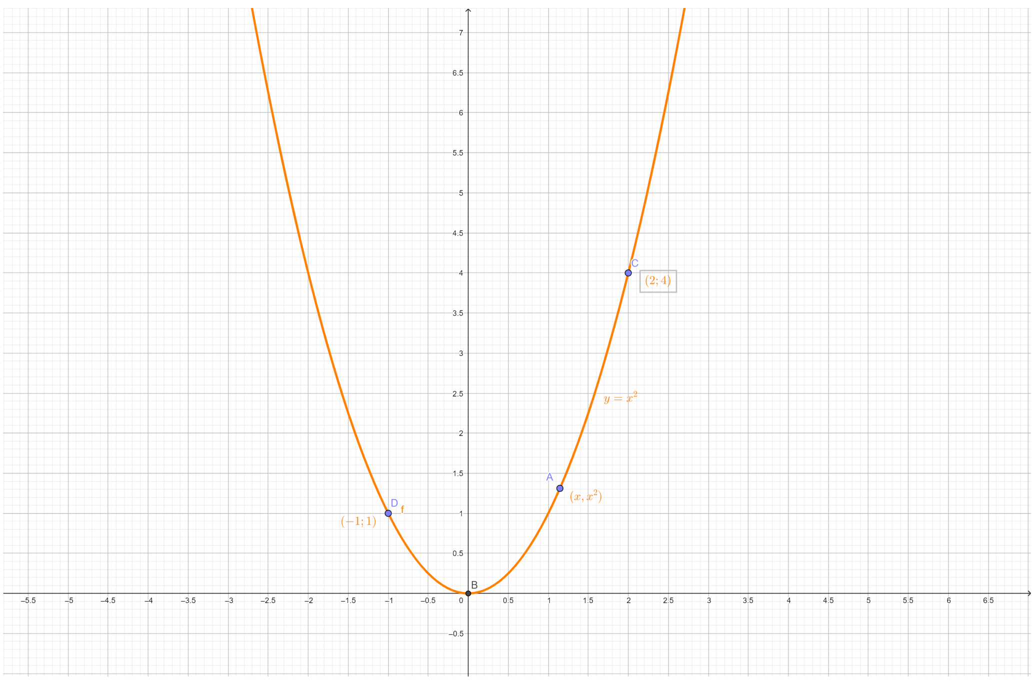
   $$(1; 2; 3); (-1; 0; 7) \in \mathbb{R}^3$$

   are two points of the space.

3. If $A := \{1; 2\}$ and $B := \{-1; 3; 7\}$, then
$$A \times B = \{(1; -1); (1; 3); (1; 7); (2; -1); (2; 3); (2; 7)\}$$
and
$$B \times A = \{(-1; 1); (-1; 2); (3; 1); (3; 2); (7; 1); (7; 2)\}.$$

4. As we can see here, usually:
$$A \times B \neq B \times A,$$
so the Cartesian product is *not commutative*.

5. The following set consists of those points of the plane, where the second component is the square of the first component, which takes as values all the real numbers:
$$P := \{(x; x^2) \in \mathbb{R}^2 \;\Big|\; x \in \mathbb{R}\}.$$
We know, that this set consists all the points of the parabola defined by the equation:
$$y = x^2 \;\; (x \in \mathbb{R}),$$
or the points of the graph of this *function*. We can also say, that by giving a rule, for example (in this case):
$$y = x^2$$
between the components of $(x; y)$ we mark out some points from $\mathbb{R}^2$. These points give us the set:
$$P = \{(x; y) \in \mathbb{R}^2 \mid y = x^2 \;\; (x \in \mathbb{R})\} \subseteq \mathbb{R}^2.$$
We can also say, that between the two components x and y of the points of this set there is a connection, a relation: namely $y = x^2$. See the lower illustration :



Let us give now the definition of a general *relation*.

Relations

Definition: Consider the nonempty sets A and B. All the nonempty subsets

$$\emptyset \neq R \subseteq A \times B$$

are called *relations*. The elements of R are ordered pairs, whose components are said to be in relation R. So if $(x; y) \in R$, then we say that x is in relation R with y. In the previous example P is a parabola and for the point $(x; y) \in P$ we have: the first component x is in relation P with y means that the second component is the square of the first one.

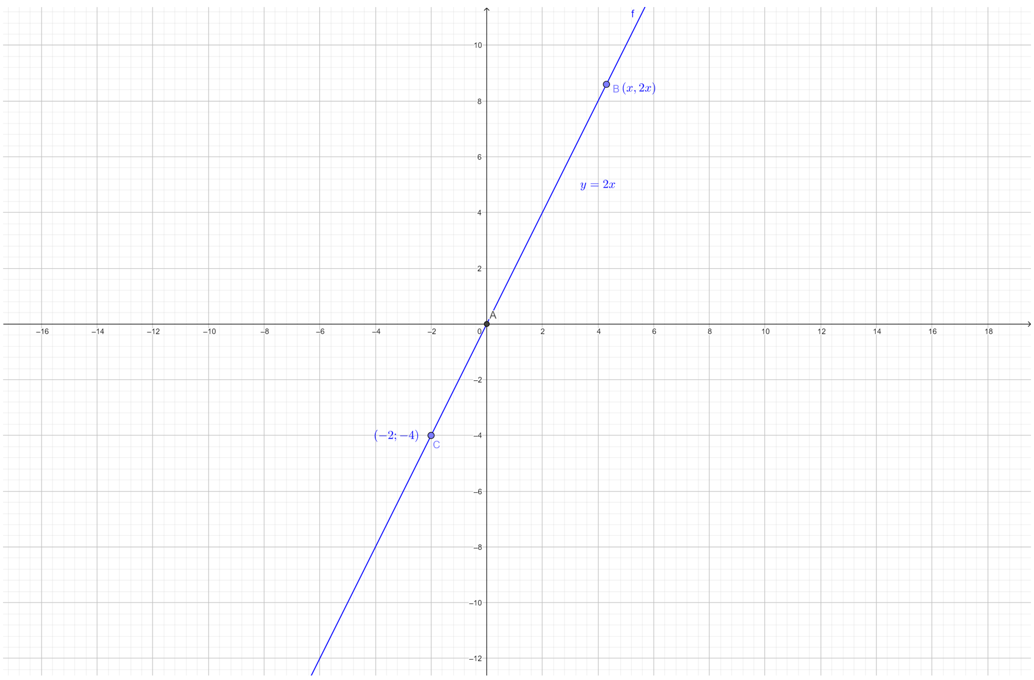
Remarks:

1. Let $A, B := \mathbb{N}$ and

$$R := \left\{(n; m) \in \mathbb{N}^2 \;\middle|\; m = 2n \;\; (n \in \mathbb{N})\right\}.$$

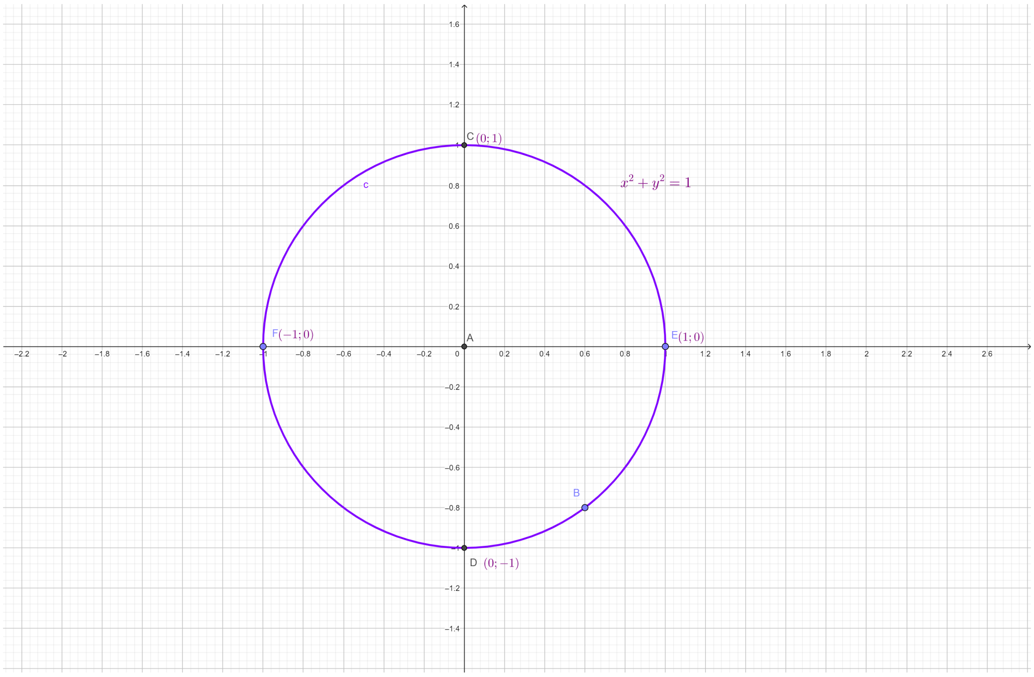
   Here an ordered pair $(n; m)$ is in relation R (or in notation $n\ R\ m$), if the second component is double the first one. Illustrate all the points of R in the plane.

2. $L := \left\{(x; y) \in \mathbb{R}^2 \;\middle|\; y = 2x \;\; (x \in \mathbb{R})\right\}$. In this case L is a line on $\mathbb{R}^2$. Illustrate all points of L in the plane. Here it is:



3. Let now $C := \left\{(x; y) \in \mathbb{R}^2 \;\middle|\; x^2+y^2 = 1\right\} \subseteq \mathbb{R}^2$. In this case C is the unit circle with center at $(0; 0)$. Observe, that: $(0; 1) \in C$ and $(0; -1) \in C$, so the first component equal to 0 is in relation C with two points 1 and -1 (two choices for $x = 0$).

Here is our set, a circle:



In the previous example of parabola P:

$$\forall\ x \in \mathbb{R}\ \exists!\ y \in \mathbb{R} :\ (x; y) \in P.$$

In this case: to all the first components x there is one and only one y, so that they are in relation P. These type of relations are called: *functions*.

$\boxed{\text{Functions}}$

Definition: Let $\emptyset \neq A, B$ be two arbitrary nonempty sets. The relation

$$\emptyset \neq f \subseteq A \times B$$

is called a *function* if:

$$\forall\ (x; y)\ \wedge\ (x; z) \in f\ \implies\ y = z.$$

Definition: Let $\emptyset \neq A, B$ be two sets and $f \subseteq A \times B$ a function.

$$D_f = \{x \in A\ \Big|\ \exists\, y \in B :\ (x; y) \in f\} \subseteq A$$

is the *domain of definition* or simply the *domain* of f , and

$$R_f = \{y \in B\ \Big|\ \exists\, x \in D_f :\ (x; y) \in f\} \subseteq B$$

is the *range of values* or simply the *range* of f.

Remarks:

1. If $f \subseteq A \times B$ is a function, then we say that f assigns to values from A (maybe not to all of them) one and only one value from B (Take care, the domain of definition for f must not be the whole set A, its only a subset of it). We will use the following notation to express this:

$$f \in A \longrightarrow B \iff f \subseteq A \times B \text{ is a function and } D_f \subseteq A.$$

When $D_f = A$, then we will use the following notation:

$$f : A \longrightarrow B,$$

which means, that $f \subseteq A \times B$ is a function and $D_f = A$.

2. Usually the range of values R_f of a function f is not the whole set B, it is only a subset of it. If $R_f = B$ we say that f is *surjective*.

3. Let $f : A \longrightarrow B$ be a function and $(x; y) \in f$ one of its ordered pairs. We say, that y is the function *value* assigned to the x and we will denote this by:

$$y = f(x).$$

Here are the usual notations to give/define a function:

$$f : A \longrightarrow B \;\; y = f(x), \;\text{ or}$$

$$y = f(x) \;\; (x \in D_f), \;\text{ or}$$

$$D_f \ni x \longmapsto y := f(x) \in B, \;\text{ or}$$

$$f := \{(x; y) \in D_f \times B \;\big|\; y := f(x)\}.$$

For example: The upper "parabola" function f can be given as follows:

$$y = x^2 \;\; (x \in \mathbb{R}), \;\text{ or}$$

$$f(x) := x^2 \;\; (x \in \mathbb{R}), \;\text{ or}$$

$$f : \mathbb{R} \longrightarrow \mathbb{R} \;\; f(x) := x^2, \;\text{ or}$$

$$f : \mathbb{R} \ni x \longmapsto x^2 \in \mathbb{R}, \;\text{ or}$$

$$f := \{(x; x^2) \in \mathbb{R}^2 \;\big|\; x \in \mathbb{R}\}, \;\text{ or}$$

$$f := \{(x; y) \in \mathbb{R}^2 \;\big|\; y = x^2\}.$$

4. Using the above notations the range of values of a function $f : A \longrightarrow B$ can be given as:
$$R_f := \{f(x) \in B \;\Big|\; x \in D_f\}.$$
For example for the function $f : \mathbb{R} \longrightarrow \mathbb{R},\;\; f(x) := x^2$ we get:
$$R_f = \{x^2 \in \mathbb{R} \;\Big|\; x \in \mathbb{R}\} = (?) = [0; +\infty).$$
The question mark above shows, that we need to prove the equality of these two sets, which will be done in the next chapter later.

5. If $A = B := \mathbb{R}$, then the functions $f \in \mathbb{R} \longrightarrow \mathbb{R}$ are called *real-to-real* or *real-real* functions.

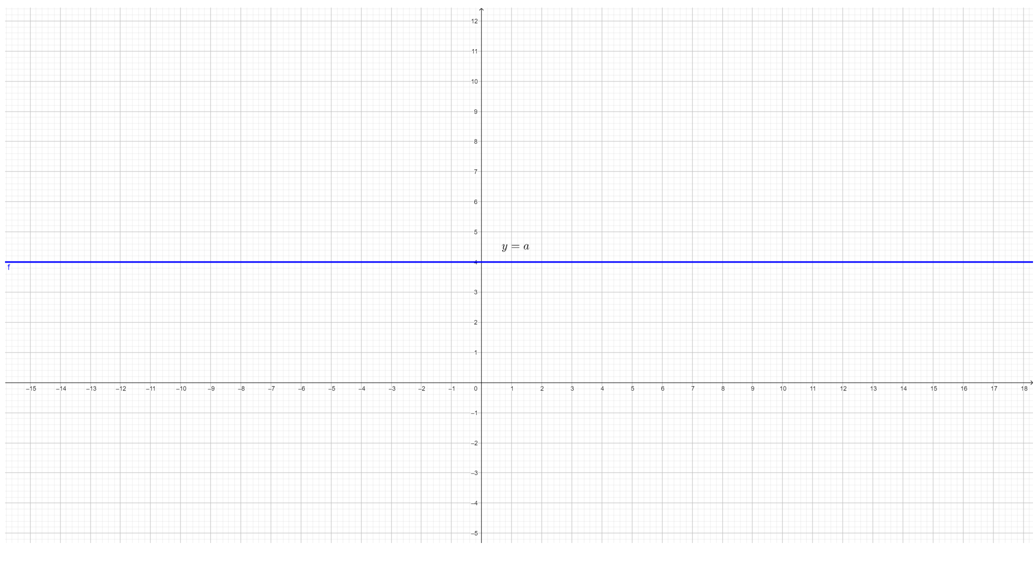
6. Let $f \in \mathbb{R} \longrightarrow \mathbb{R}$ be a real-real function. The following set in $\mathbb{R}^2$ is called the *graph* of f:
$$\text{graph}(f) := \Big\{(x; f(x)) \in \mathbb{R}^2 \;\Big|\; x \in D_f\Big\} \subset \mathbb{R}^2.$$

Some basic functions

Here is a list of some basic functions we will need later, so we recommend to everybody to review/study them.

1. Constant functions: with a fixed $a \in \mathbb{R}$:
$$f(x) := a \;\; (x \in \mathbb{R}).$$

Here is the graph:



2. Linear functions (first order polynomials/lines): with $a, b \in \mathbb{R}, a \neq 0$:

$$f(x) := ax + b \ \ (x \in \mathbb{R}).$$

3. Quadratic functions (second order polynomials/parabolas): with $a, b, c \in \mathbb{R}, a \neq 0$:

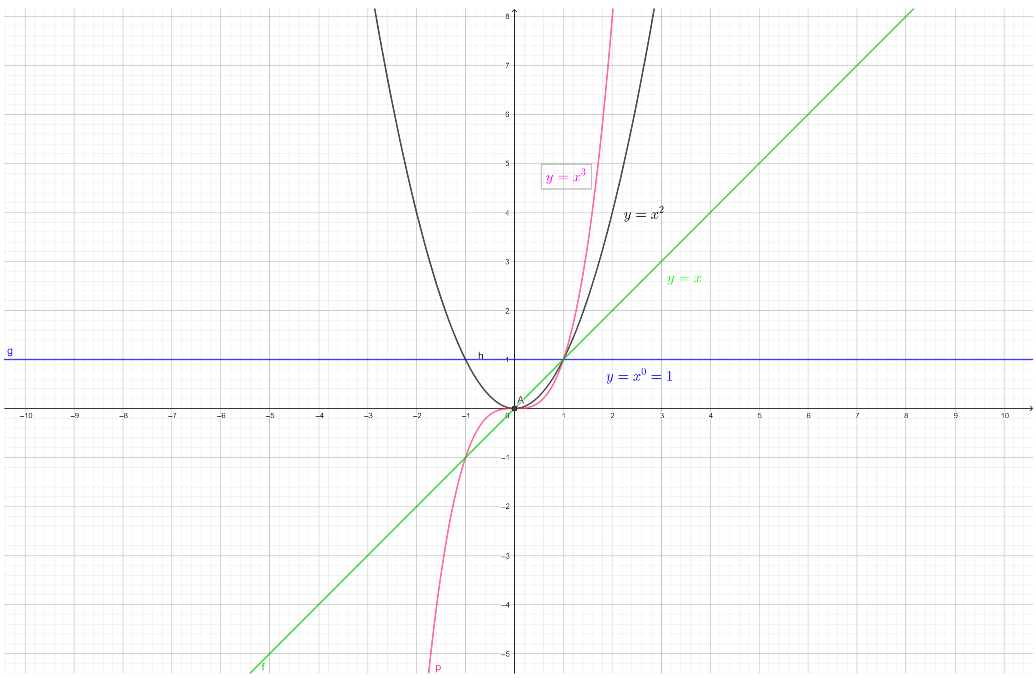
$$f(x) := ax^2 + bx + c \ \ (x \in \mathbb{R}).$$

4. Power functions: with $n \in \mathbb{N}$:

$$f(x) := x^n \ \ (x \in \mathbb{R}).$$

   Especially the cases when $n = 0, 1, 2, 3$.

   Here are the graphs for the given special n values:



5. Root-functions: fix an $1 \leq n \in \mathbb{N}$ and

$$f(x) := \sqrt[2n]{x} \ \ (x \in [0; +\infty)) \ \ \wedge \ \ g(x) := \sqrt[2n+1]{x} \ \ (x \in \mathbb{R}).$$

   Especially the cases when $n = 1$: the square-root and the cube-root functions.

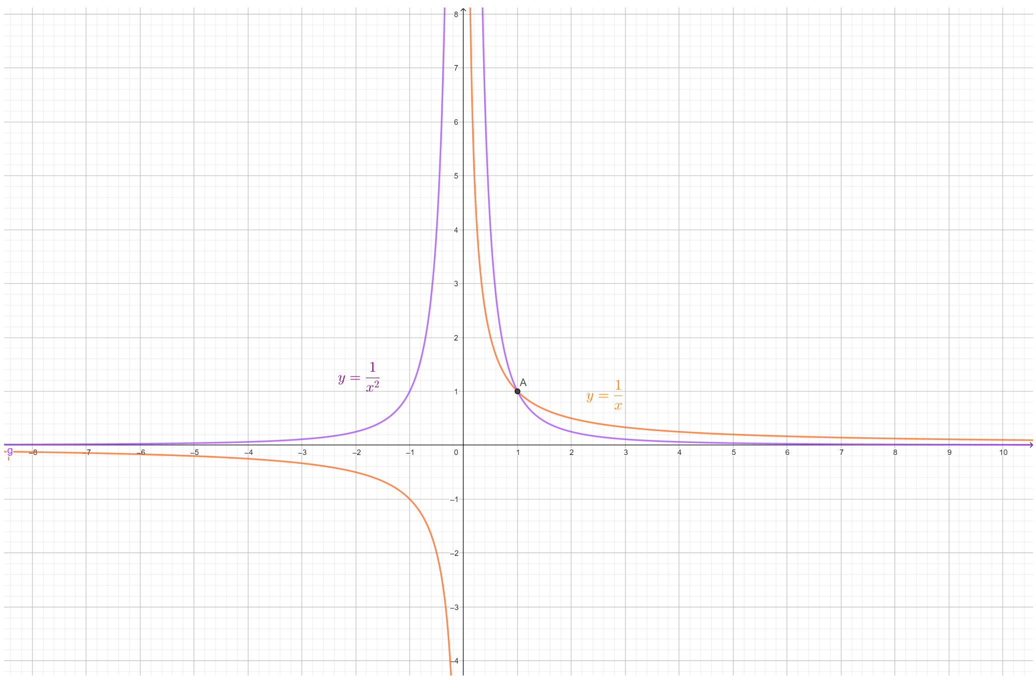
Here are the graphs:

6. Some special fraction functions, namely with an $1 \leq n \in \mathbb{N}$:

$$f(x) := \frac{1}{x^n} \ (x \in \mathbb{R} \setminus \{0\}).$$

Especially the cases when $n = 1, 2$.

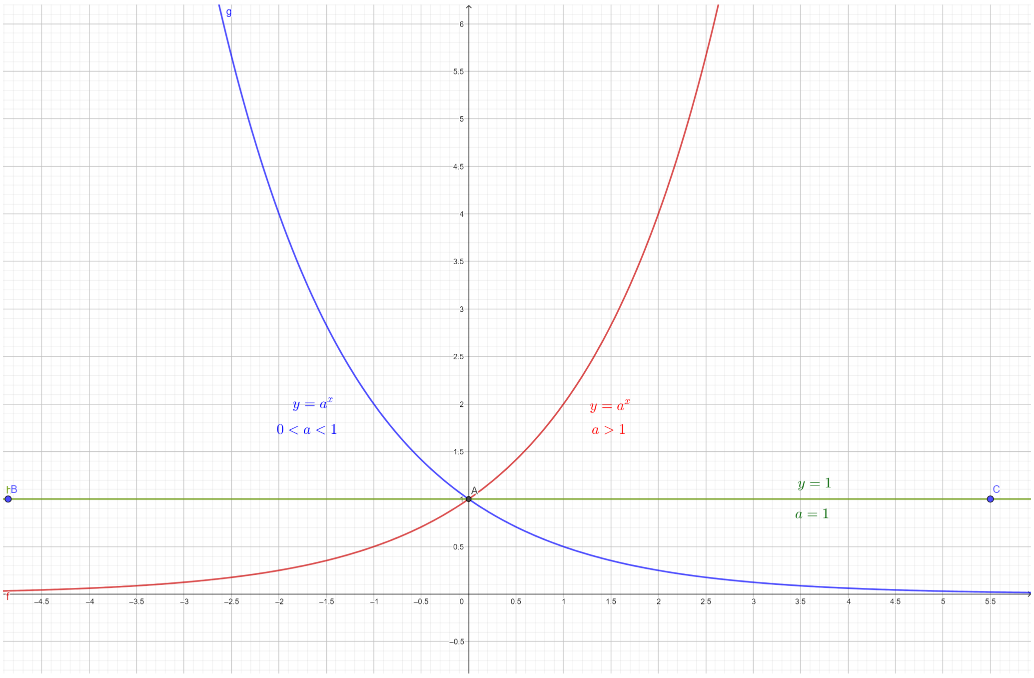
See their graphs below:



7. Exponential functions: with base $0 < a \neq 1$:

$$f(x) := a^x \quad (x \in \mathbb{R}).$$

See their graphs below:



Special case whith $a = e$:

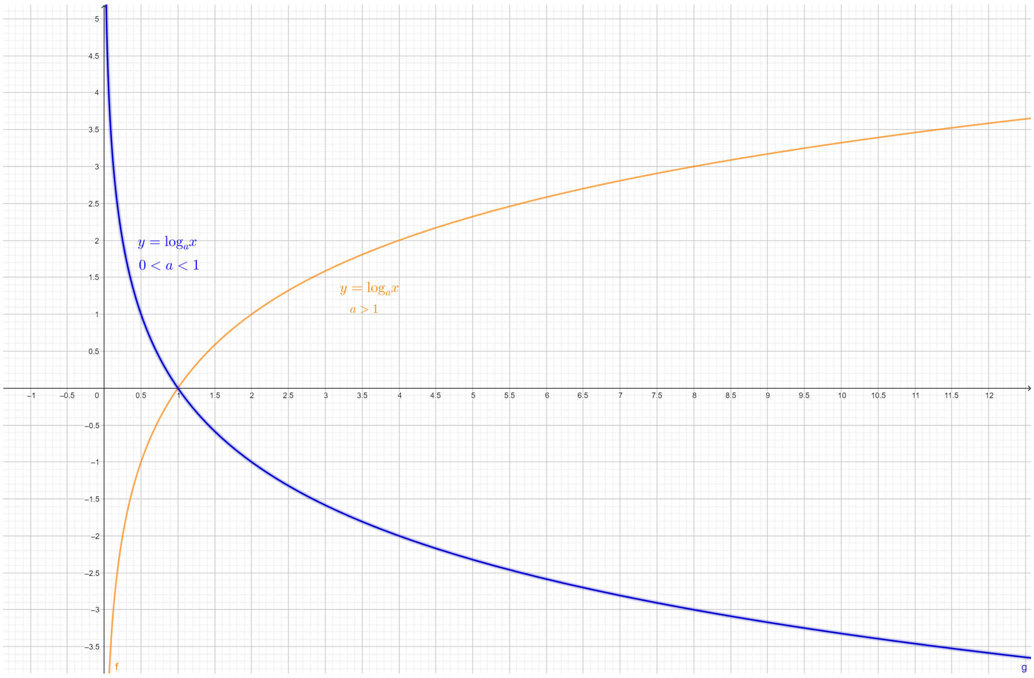
$$f(x) := e^x \quad (x \in \mathbb{R}).$$

See its graph below:

8. Logarithmic functions: with base $0 < a \neq 1$:

$$f(x) := \log_a(x) \ \ (x \in (0; +\infty)).$$

See their graphs below:



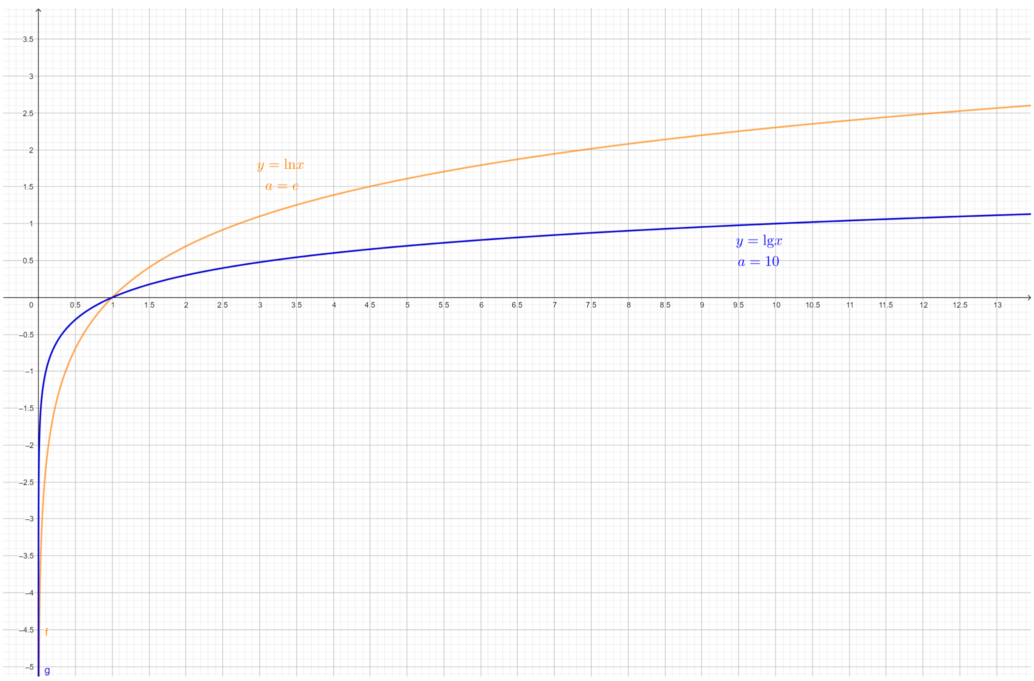
Special case, when $a = e$:

$$f(x) := \ln x := \log_e(x) \ \ (x \in (0; +\infty)).$$

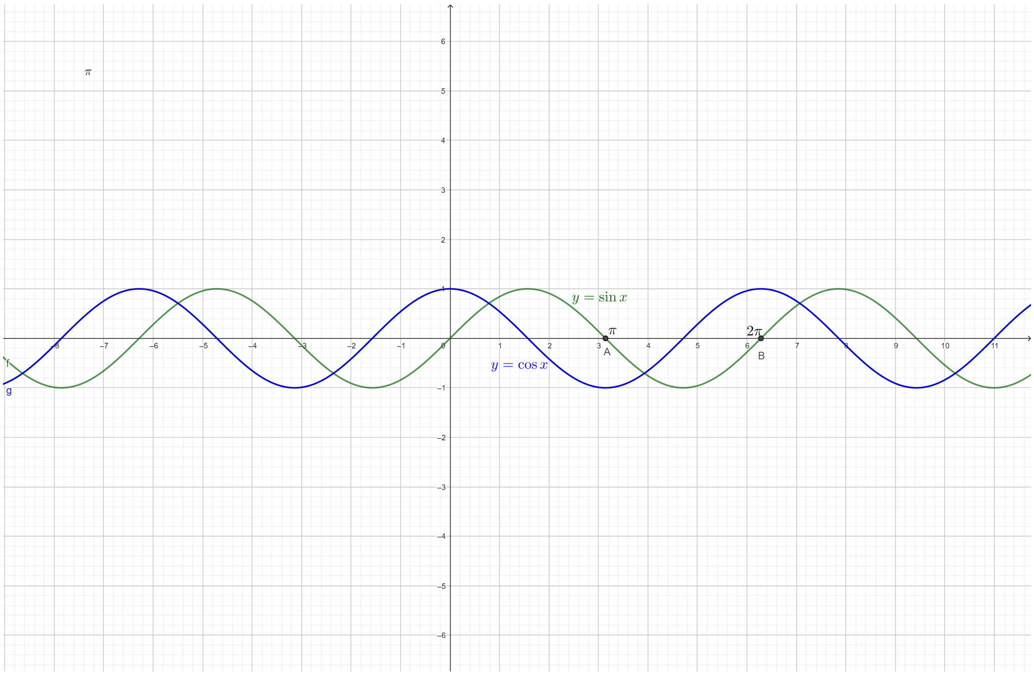
Another important case is when $a = 10$:

$$f(x) := \lg x := \log_{10}(x) \ \ (x \in (0; +\infty)).$$

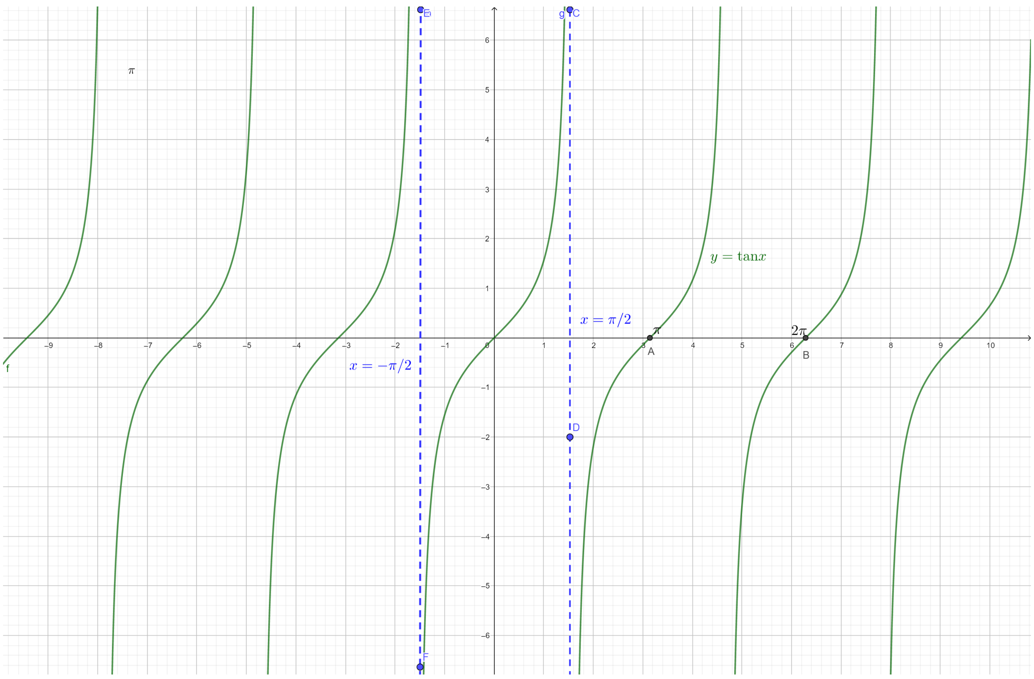
See their graphs below:



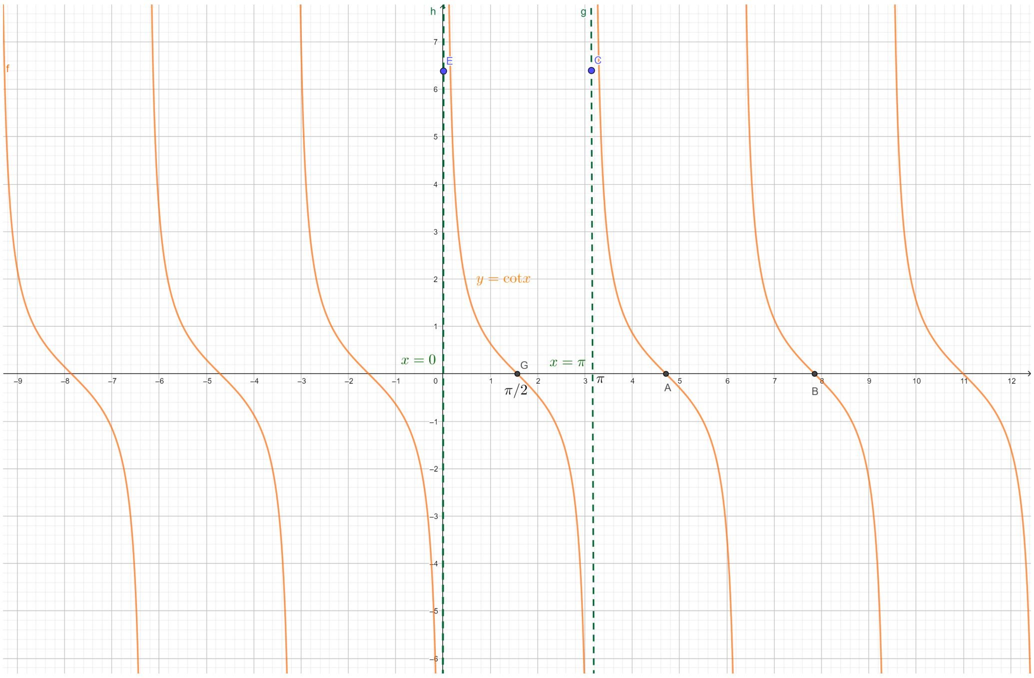
9. The basic trigonometric functions: $\sin, \cos, \tan, \cot$. See the graphs of sin and cos functions below:



See the graph of the *tan* function below:



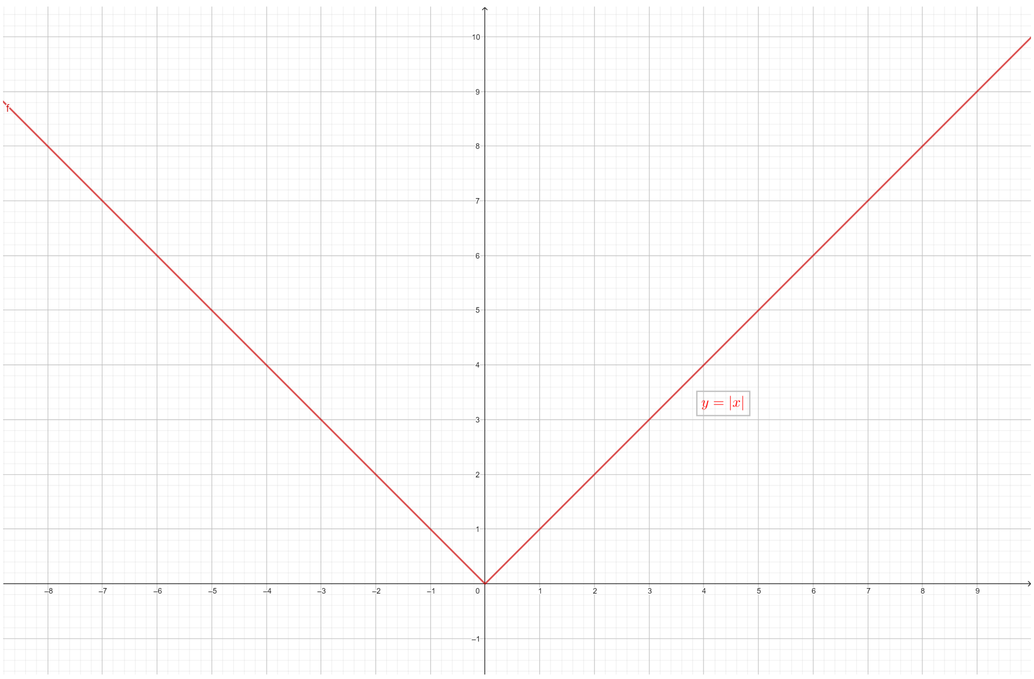
See the graph of the *cot* function here:



10. Absolute-value function:

$$f(x) := \mathrm{Abs}\,(x) := |x| = \begin{cases} -x, & \text{if } x < 0; \\ x, & \text{if } x \geq 0. \end{cases}$$

See its graph below:



11. Integer-part function or floor-function:

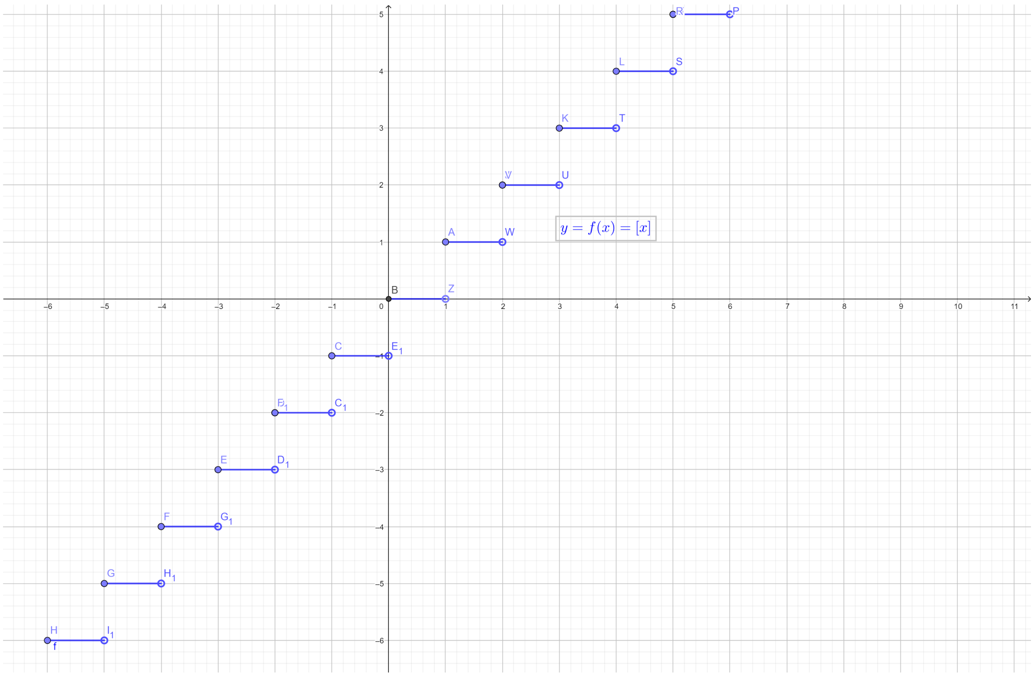
$$f(x) := [x] \ \ (x \in \mathbb{R});$$

where $[x] :=$ is the closest integer to x which is not greater then x. For example:

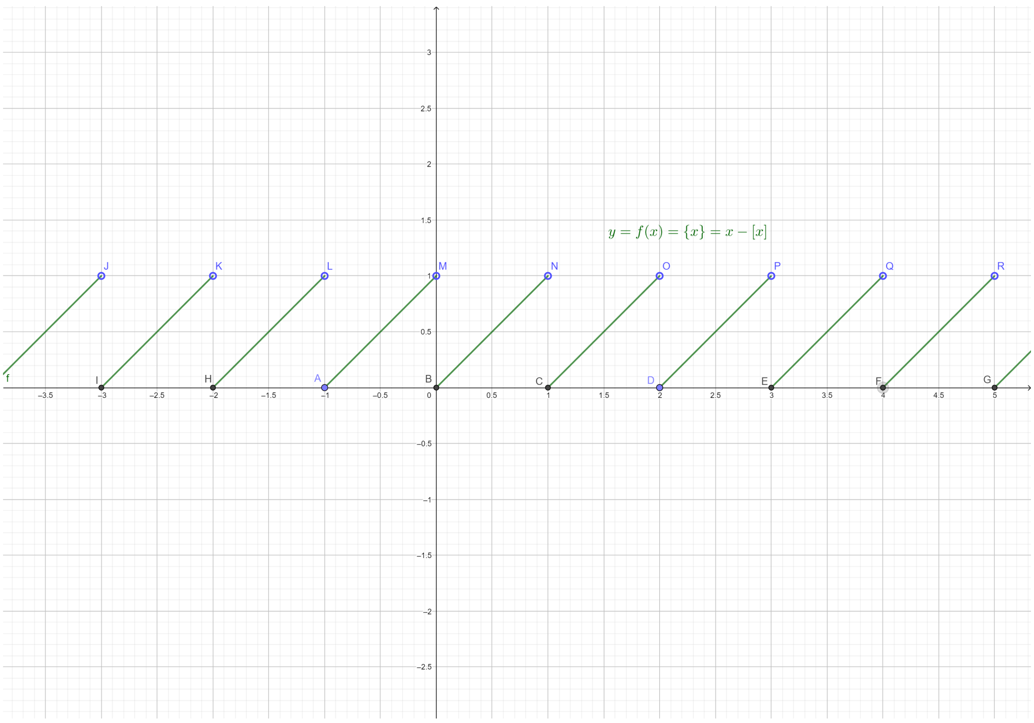
$$[3.75] = 3; \ \ [-1.22] = -2; \ \ [1.22] = 1; \ \ [-0.176] = -1; \ \ [2] = 2;$$

$$[\sqrt{2}] = 1; \ \ [e] = 2; \ \ [\pi] = 3; \ \ [5/3] = 1.$$

See its graph below:



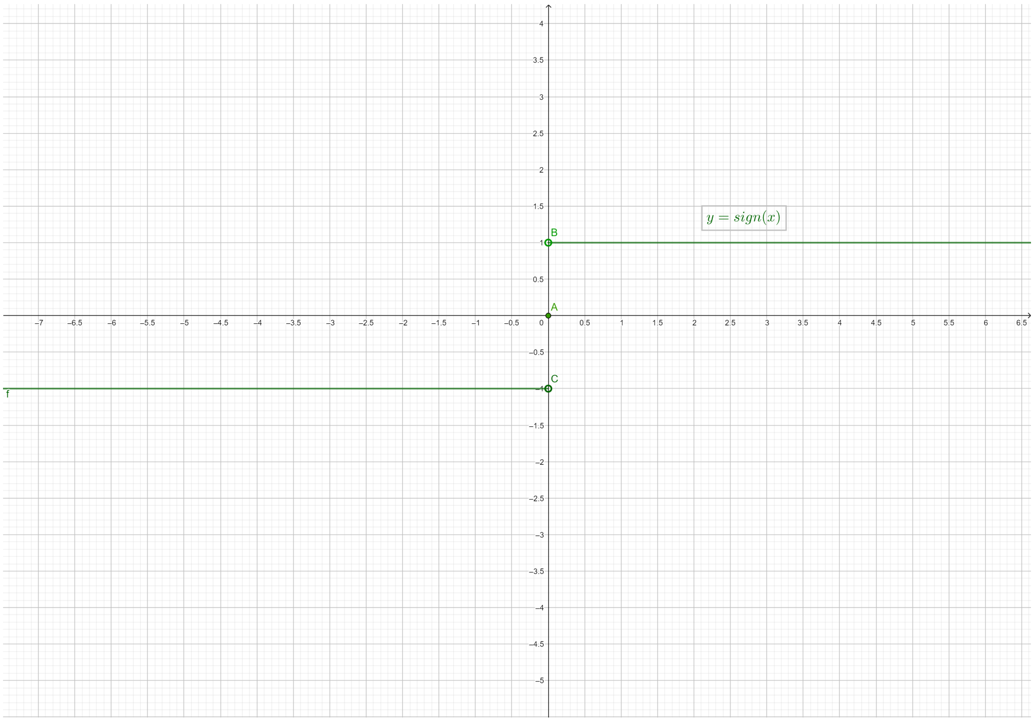
12. Fractional-part function: $f(x) := \{x\} := x - [x]$ $(x \in \mathbb{R})$. See its graph below:



13. Sign/Signum function:

$$f(x) := \text{Sign}(x) := \begin{cases} -1, & \text{if } x < 0; \\ 0, & \text{if } x = 0; \\ 1, & \text{if } x > 0. \end{cases}$$

See its graph below:



14. The Dirichlet function:

$$D(x) := \begin{cases} 1, & \text{if } x \in \mathbb{R} \cap \mathbb{Q}; \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Operations with functions

Equality of functions

Definition: Consider the sets $\emptyset \neq A, B, C, D$ and the functions $f \in A \longrightarrow B$ and $g \in C \longrightarrow D$. We say, that $f = g$ (the two functions are equal, or they are the same) if and only if:

$$D_f = D_g \ \wedge \ (\forall \ x \in D_f = D_g : f(x) = g(x)).$$

For example:

1. If

$$f(x) = \sqrt{x^3} \ \ (x \in [0; +\infty)); \ \ g(x) = x \cdot \sqrt{x} \ \ (x \in [0; +\infty)),$$

then we have $D_f = D_g = [0; +\infty)$ and

$$f(x) = |x| \cdot \sqrt{x} = (x \geq 0) = x \cdot \sqrt{x} = g(x) \ \ \forall \ x \in [0; +\infty) \Longrightarrow f = g.$$

2. When
$$f(x) = \sqrt{x^2} \ \ (x \in \mathbb{R}); \ \ g(x) = x \ \ (x \in \mathbb{R}),$$
then $D_f = D_g = \mathbb{R}$, but $f(x) = \sqrt{x^2} = |x| \neq g(x) = x$, if $x \in (-\infty; 0) \Longrightarrow f \neq g$.

3. Consider now
$$f(x) = x - 1 \quad (x \in \mathbb{R}); \ \ g(x) = \frac{x^2 - 1}{x + 1} \ \ (x \in \mathbb{R} \setminus \{-1\}).$$
In this case when $-1 \neq x \in \mathbb{R}$, then $g(x) = \dfrac{(x-1) \cdot (x+1)}{x+1} = x - 1 = f(x)$, but
$$D_f = \mathbb{R} \neq D_g = \mathbb{R} \setminus \{-1\} \Longrightarrow f \neq g.$$

$\boxed{\text{Arithmetic operations with functions}}$

Definition: Consider the sets $\emptyset \neq A, B$ and the functions
$$f \in A \longrightarrow \mathbb{R} \text{ and } g \in B \longrightarrow \mathbb{R}$$
and $c \in \mathbb{R}$ a fixed real number. We can define the following new functions (supposing, that the given domains of definitions are not empty):
$$D_{c \cdot f} := D_f \ \ \wedge \ \ (c \cdot f)(x) := c \cdot f(x) \ \ (\forall\, x \in D_{c \cdot f});$$
$$D_{f+g} := D_f \cap D_g \neq \emptyset \ \ \wedge (f+g)(x) := f(x) + g(x) \ \ (\forall\, x \in D_{f+g});$$
$$D_{f-g} := D_f \cap D_g \neq \emptyset \ \ \wedge (f-g)(x) := f(x) - g(x) \ \ (\forall\, x \in D_{f-g});$$
$$D_{f \cdot g} := D_f \cap D_g \neq \emptyset \ \ \wedge (f \cdot g)(x) := f(x) \cdot g(x) \ \ (\forall\, x \in D_{f \cdot g});$$
$$D_{f/g} := \{x \in D_f \cap D_g \ \Big| \ g(x) \neq 0\} \neq \emptyset \ \ \wedge \ \ \frac{f}{g}(x) := \frac{f(x)}{g(x)} \ \ (\forall\, x \in D_{f/g}).$$
The given functions are the following:

1. $c \cdot f$ the *scaled* or *constant-times-function*,
2. $f + g$ the *sum-function*,
3. $f - g$ the *difference-function*,
4. $f \cdot g$ the *product-function*,
5. f/g the *quotient-function*.

Examples:

1. Consider the following functions:

$$f(x)=\sqrt{x-1}\ \ (x\in[1;+\infty));\ \ g(x)=\sin x\ \ (x\in[0;2\pi]).$$

We have now:

$$D_{2\cdot f}=[1;+\infty)\ \ \wedge\ \ (2\cdot f)(x)=2\cdot f(x)=2\cdot\sqrt{x-1}\ \ (x\in[1;+\infty));$$

$$D_{f+g}=[1;+\infty)\cap[0;2\pi]=[1;2\pi]\ \ \wedge\ \ (f+g)(x):=f(x)+g(x)=\sqrt{x-1}+\sin x;$$

$$D_{f-g}=[1;+\infty)\cap[0;2\pi]=[1;2\pi]\ \ \wedge\ \ (f-g)(x):=f(x)-g(x)=\sqrt{x-1}-\sin x;$$

$$D_{f\cdot g}=[1;+\infty)\cap[0;2\pi]=[1;2\pi]\ \ \wedge\ \ (f\cdot g)(x):=f(x)\cdot g(x)=\sqrt{x-1}\cdot\sin x;$$

$$D_{f/g}=\left\{x\in[1;2\pi]\ \middle|\ \sin x\neq 0\right\}=[1;\pi)\cup(\pi;2\pi)\ \ \wedge\ \ \frac{f}{g}(x):=\frac{f(x)}{g(x)}=\frac{\sqrt{x-1}}{\sin x};$$

$$D_{g/f}=\left\{x\in[1;2\pi]\ \middle|\ \sqrt{x-1}\neq 0\right\}=(1;2\pi]\ \ \wedge\ \ \frac{g}{f}(x):=\frac{g(x)}{f(x)}=\frac{\sin x}{\sqrt{x-1}}.$$

Compositions of functions

Definition: Consider the sets $\emptyset\neq A,B,C,D$ and the functions

$$f\in A\longrightarrow B \text{ and } g\in C\longrightarrow D.$$

We can define the new function $f\circ g$, the *composition* of f and g (in this order) as follows:

$$D_{f\circ g}:=\left\{x\in D_g\ \middle|\ g(x)\in D_f\right\}\neq\emptyset\ \ \wedge\ \ (f\circ g)(x):=f(g(x))\ \ (x\in D_{f\circ g}).$$

For example:

1. Find $f\circ g$ for the above given functions

$$f(x)=\sqrt{x-1}\ \ (x\in[1;+\infty));\ \ g(x)=\sin x\ \ (x\in[0;2\pi]).$$

$$D_{f\circ g}=\left\{x\in[0;2\pi]\ \middle|\ \sin x\in[1;+\infty)\right\}=(\star)=\left\{\frac{\pi}{2}\right\}\ \wedge$$
$$\wedge\ (f\circ g)(x):=f(g(x))=\sqrt{\sin x-1}\ \ \left(x\in\left\{\frac{\pi}{2}\right\}\right).$$

Here $(\star)$ stays for solving the following equation on interval $[0;2\pi]$:

$$\sin x\in[1;+\infty)\ \iff\ \sin x=1\ \iff\ x=\frac{\pi}{2}+2k\pi\ \ (k\in\mathbb{Z}).$$

From these values only $\pi/2$ is in the interval $[0;2\pi]$, so $f\circ g$ can be defined in only one point:

$$D_{f\circ g}=\{\pi/2\}\ \ \wedge\ \ (f\circ g)(\pi/2)=\sqrt{\sin(\pi/2)-1}=0.$$

In this case the graph of $f\circ g$ consists of only one point, namely $(\pi/2;0)$.

2. Lets evaluate the composition in reversed order $g \circ f$:

$$D_{g\circ f} = \{x \in D_f \;\Big|\; f(x) \in D_g\} = \{x \in [1; +\infty) \;\Big|\; \sqrt{x-1} \in [0; 2\pi]\} = (\star) = [1; 1{+}4\pi^2] \;\wedge$$

$$\wedge\; (g \circ f)(x) = g(f(x)) = \sin(\sqrt{x-1}) \;\; (x \in [1; 1+4\pi^2]),$$

where $(\star)$ denotes again the solving process of the upper inequalities on $[1; +\infty)$:

$$\sqrt{x-1} \in [0; 2\pi] \iff 0 \le \sqrt{x-1} \le 2\pi \iff 0 \le x-1 \le 4\pi^2 \iff$$

$$\iff 1 \le x \le 1 + 4\pi^2.$$

Taking the intersection with $[1; +\infty)$ we get the set $[1; 4\pi^2]$.

3. **Remark:** as we could see here, it is not alway true that:

$$f \circ g = g \circ f,$$

which means, that the composition of functions is not commutative (there are cases when it is, but usually not).

$\boxed{\text{Transformations of functions}}$

Here are some basic transformations we will consider to be known.

1. The transformations of the argument x

   (a) Shifting the variable x along the $x-$axes with value $a \in \mathbb{R}$ (the graph of f is shifted with a along the $x-$line to left when $a > 0$ and to right when $a < 0$):

   $$g(x) := f(x+a) \;\; (x + a \in D_f);$$

   $$g(x) := f(x-a) \;\; (x - a \in D_f).$$

   (b) Mirroring to the $y-$axes (take the reflection of the graph of f to the $y-$axes):

   $$g(x) := f(-x) \;\; (-x \in D_f).$$

   (c) Dilatation along x with a constant $a > 0$:

   $$g(x) := f(a \cdot x) \;\; (a > 0 \;\wedge\; a \cdot x \in D_f).$$

   Remarks: When $a < 0$ we can do first the multiplication by $-a$ (see the positive case) and after this we take a reflection to $y-$axes. In both cases the "measure" of the dilatation (related to the x axes) is $1/|a|$.

   (d) "Cut and reflect" (We omit/cut off the $x < 0$ part of the graph of f and the right-hand side of it (when $x \ge 0$) gets reflected on the y axes):

   $$g(x) := f(|x|) = \begin{cases} f(-x) & \text{, if } x < 0; \\ f(x) & \text{, if } x \ge 0; \end{cases} \qquad (|x| \in D_f).$$

2. Transformations of the function values y

   (a) Shifting the values of the function along the $y-$axes with a constant $c \in \mathbb{R}$ (we shift the values of f with c along the $y-$axes , if $c > 0$ upwards, if $c < 0$ downwards):

   $$g(x) := f(x) + c \quad (x \in D_f);$$
   $$g(x) := f(x) - c \quad (x \in D_f).$$

   (b) Reflecting/mirroring the graph of f to the $x-$line:

   $$g(x) := -f(x) \quad (x \in D_f).$$

   (c) Dilatation of the values along the $y-$axes with the constant $c > 0$:

   $$g(x) := c \cdot f(x) \quad (c > 0 \ \wedge \ x \in D_f).$$

   Remark: When $c < 0$ first we make the multiplication by $-c > 0$ and then we mirror all the points of the graph to the $x-$line.

   (d) Cutting and mirroring:

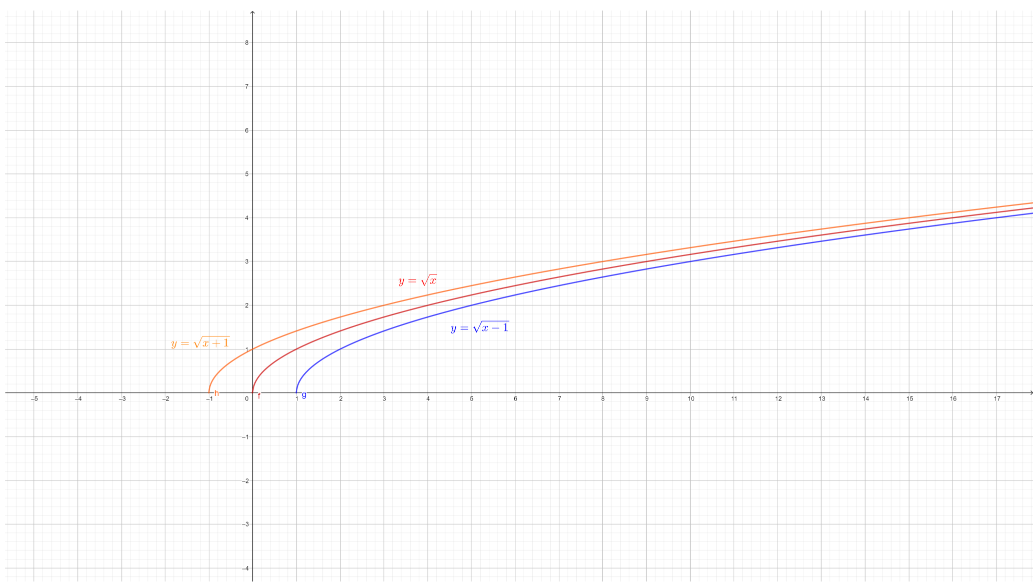
   $$g(x) := |f(x)| = \begin{cases} -f(x) \text{ , if } f(x) < 0; \\ f(x) \quad \text{ , if } f(x) \geq 0; \end{cases} \quad (x \in D_f).$$

For example: As a good practice for your own, sketch the graphs of the following functions (starting with f) and then make all the indicated step-by-step transformations:

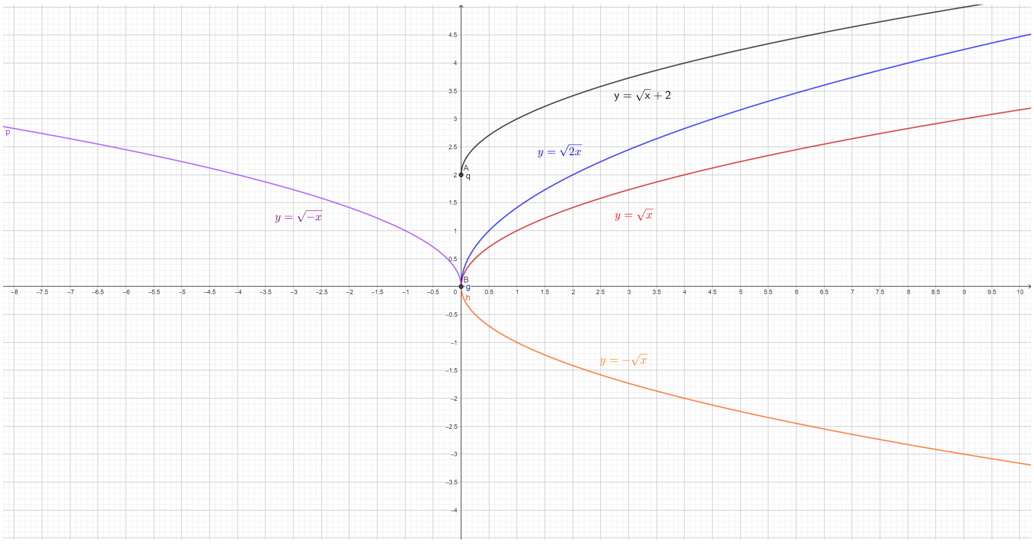
1. Consider the function $f(x) := \sqrt{x} \ (x \geq 0)$:

$$1 \leq x \longmapsto \sqrt{x-1}; \quad -1 \leq x \longmapsto \sqrt{x+1}; \quad 0 \leq x \longmapsto \sqrt{2x}; \quad 0 \leq x \longmapsto \sqrt{x} - 1;$$
$$0 \leq x \longmapsto 2 \cdot \sqrt{x}; \quad 0 \leq x \longmapsto -\sqrt{x}; \quad 0 \geq x \longmapsto \sqrt{-x}; \quad 0 \leq x \longmapsto \sqrt{x} + 2;$$
$$\mathbb{R} \ni x \longmapsto \sqrt{|x|}; \quad 0 \leq x \longmapsto |\sqrt{x}|.$$

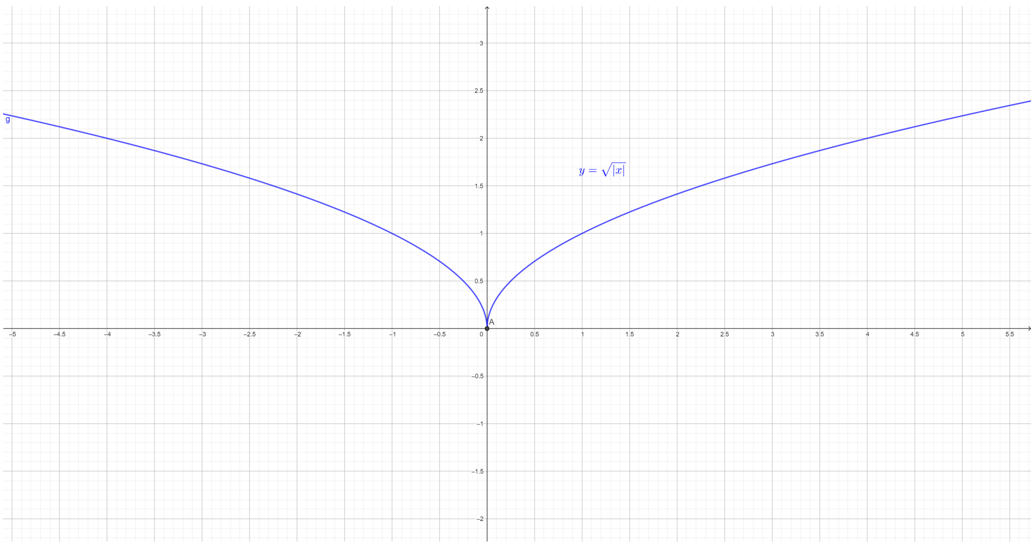
See the indicated graphs below:



See the more graphs below:



Last graph here:



2. Consider the function $f(x) := x^3 \ (x \in \mathbb{R})$:

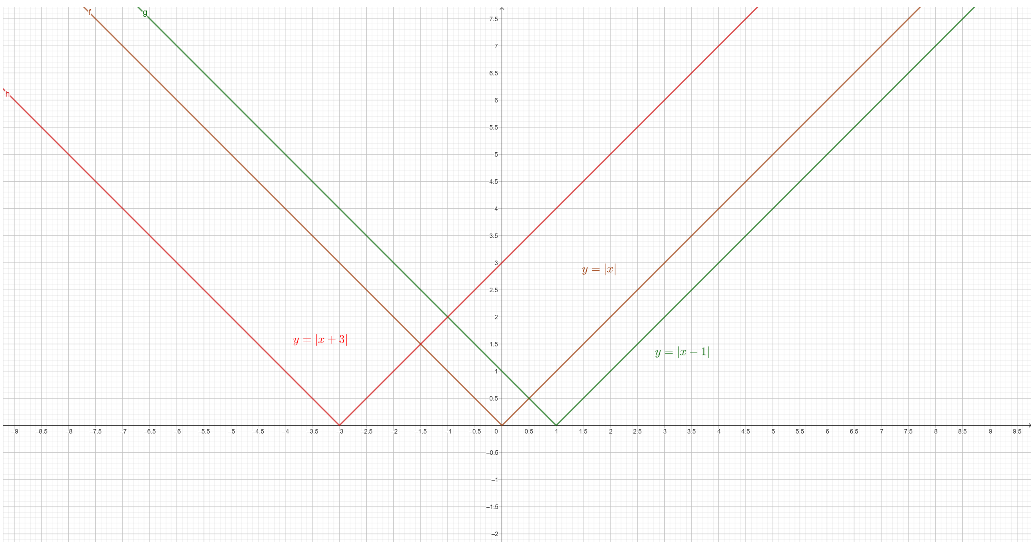
$$\mathbb{R} \ni x \longmapsto (x-2)^3; \quad \mathbb{R} \ni x \longmapsto (x+1)^3; \quad \mathbb{R} \ni x \longmapsto (2x)^3; \quad \mathbb{R} \ni x \longmapsto x^3+1;$$
$$\mathbb{R} \ni x \longrightarrow 2 \cdot x^3; \quad \mathbb{R} \ni x \longmapsto -x^3; \quad \mathbb{R} \ni x \longmapsto (-x)^3;$$
$$\mathbb{R} \ni x \longmapsto x^3 - 8; \quad \mathbb{R} \ni x \longrightarrow (|x|)^3; \quad 0 \leq x \longmapsto |x^3|.$$

3. Consider the function $f(x) := |x| \ (x \in \mathbb{R})$:

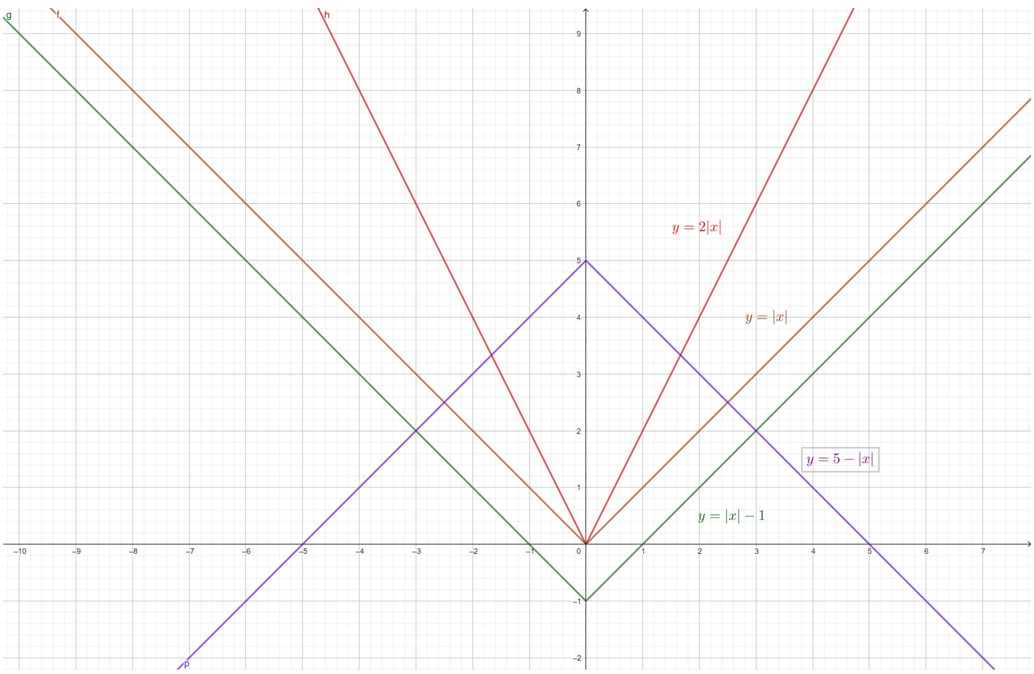
$$\mathbb{R} \ni x \longrightarrow |x-2|; \quad \mathbb{R} \ni x \longmapsto |x+1|; \quad \mathbb{R} \ni x \longmapsto |2x|; \quad \mathbb{R} \ni x \longmapsto |x|-1;$$

$$\mathbb{R} \ni x \longrightarrow 2 \cdot |x|; \quad \mathbb{R} \ni x \longmapsto -|x|; \quad \mathbb{R} \ni x \longmapsto |-x|; \quad \mathbb{R} \ni x \longmapsto |x|+2;$$
$$\mathbb{R} \ni x \longrightarrow ||x|-1|.$$

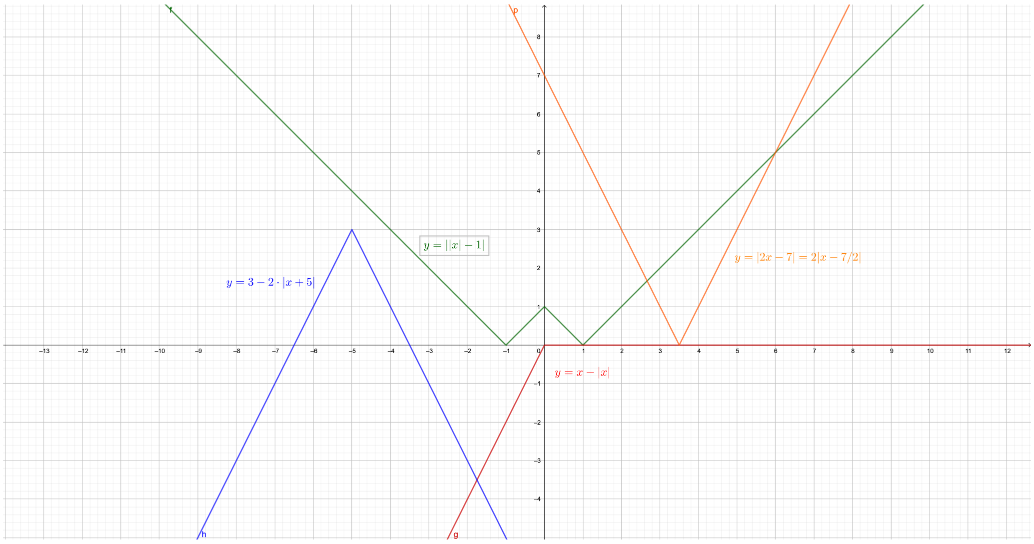
See some transformations and their graphs below:



See more graphs below:



Some more transformations and graphs:



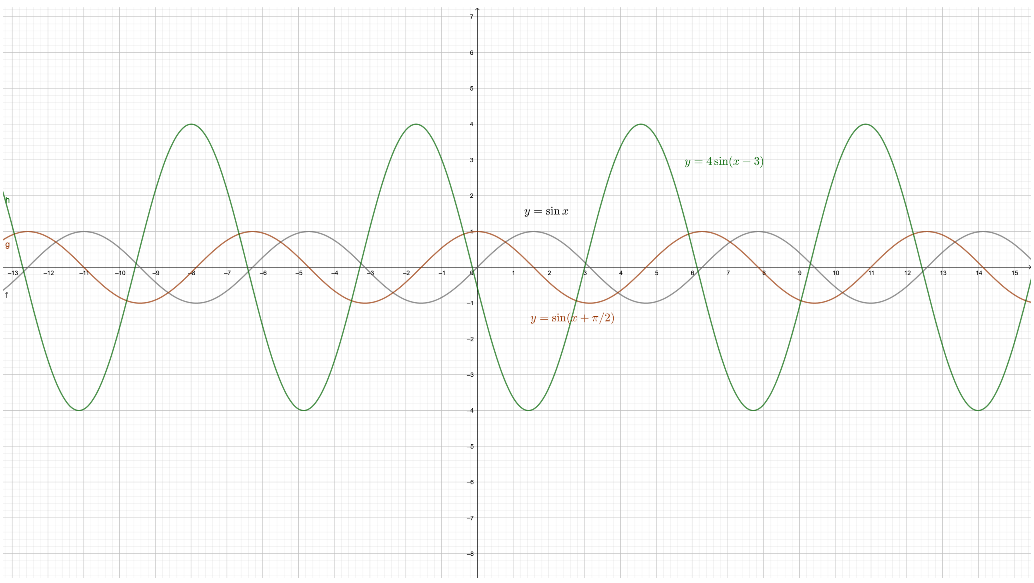
4. Consider the function $f(x) := \sin x \ \ (x \in \mathbb{R})$:

$$\mathbb{R} \ni x \longrightarrow \sin(x - \pi/3); \quad \mathbb{R} \ni x \longmapsto \sin(x + \pi/2); \quad \mathbb{R} \ni x \longmapsto \sin(2x);$$

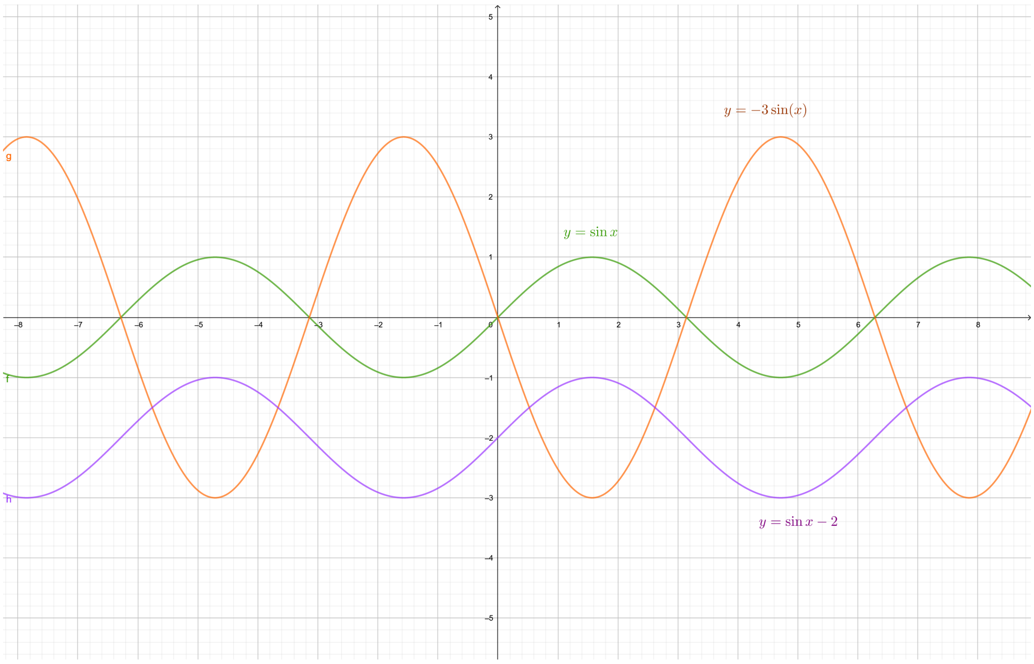
$$\mathbb{R} \ni x \longrightarrow 2 \cdot \sin x; \quad \mathbb{R} \ni x \longmapsto -\sin x; \quad \mathbb{R} \ni x \longmapsto \sin(-x); \quad \mathbb{R} \ni x \longrightarrow \sin x - 2;$$

$$\mathbb{R} \ni x \longrightarrow 1 + \sin x; \quad \mathbb{R} \ni x \longrightarrow |\sin x|; \quad \mathbb{R} \ni x \longrightarrow \sin(|x|).$$

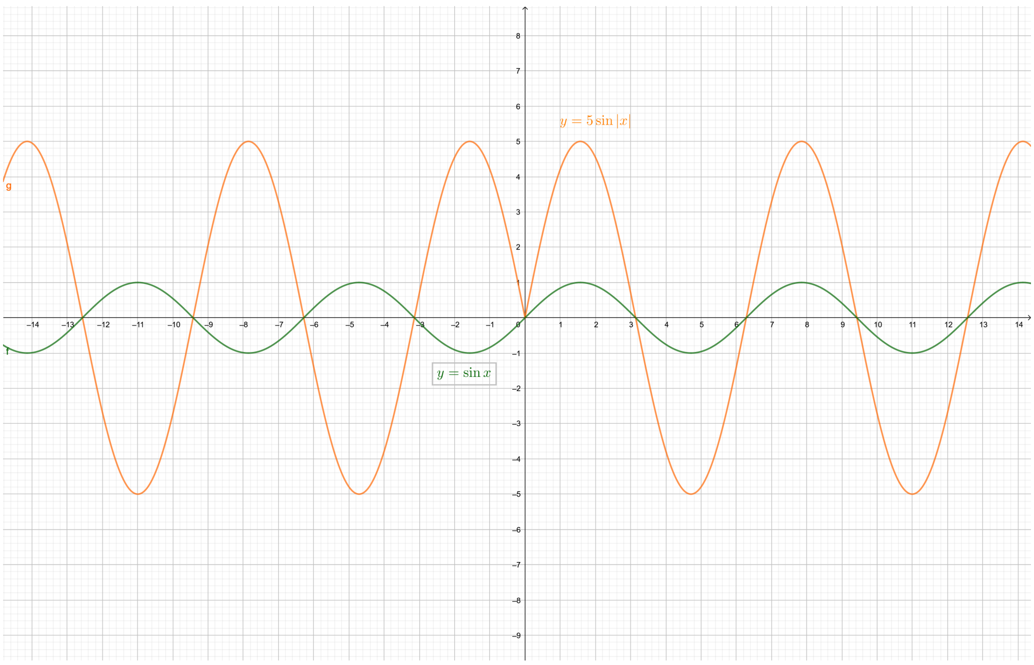
See some transformations and their graphs below:



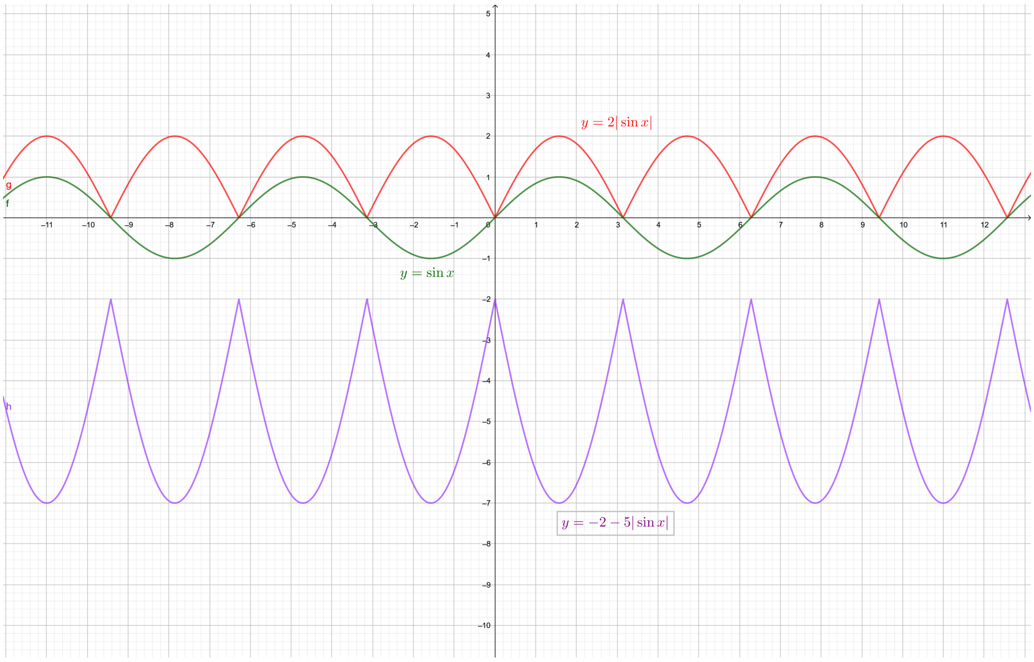
See more graphs below:



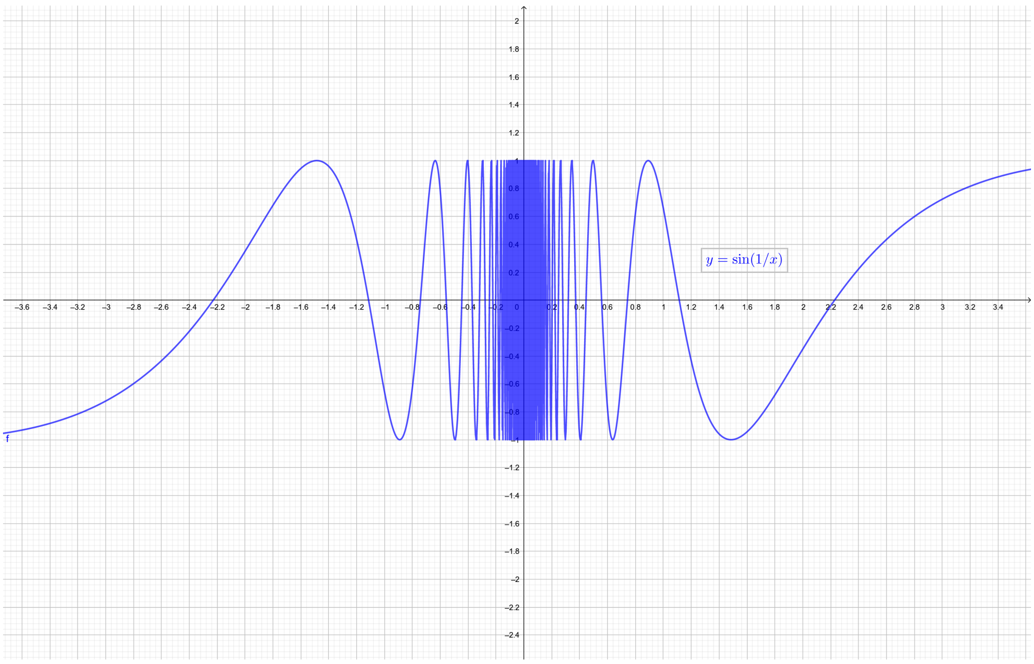
Some more transformations and graphs:



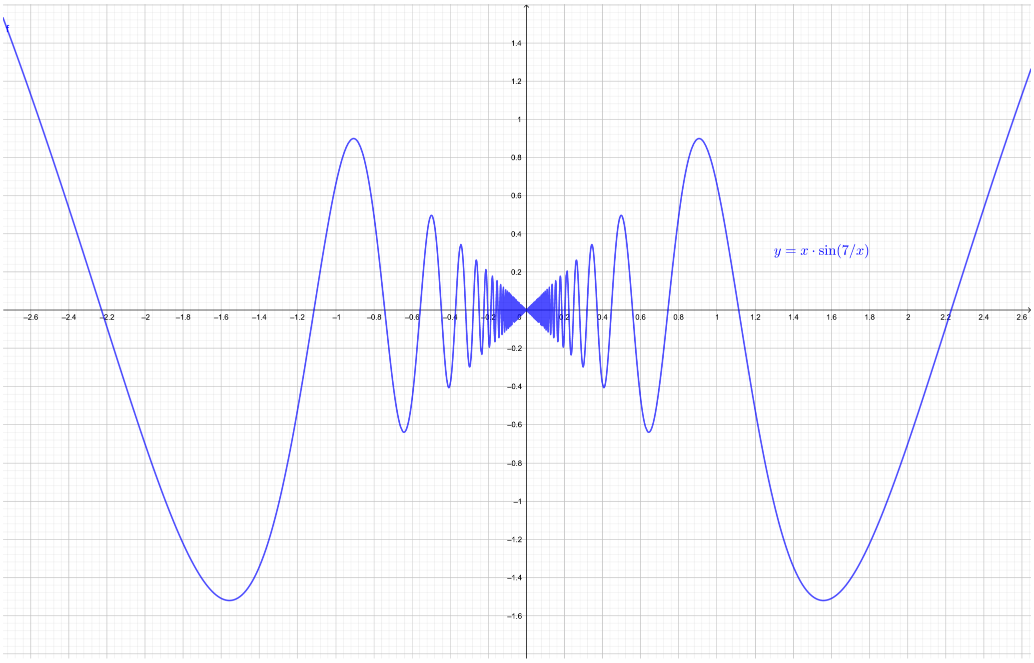
One more transformation:



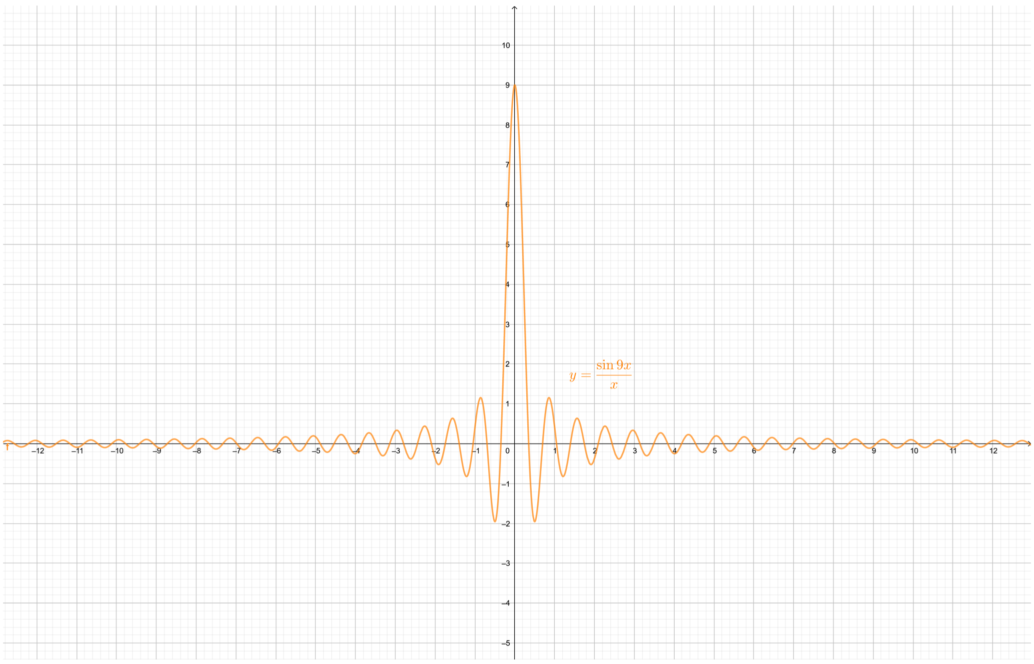
Some interesting graphs:



Try to see, why the following graph looks like this:



Try to find out, why the following graph looks like this:



5. Consider the function $f(x) := \cos x \ \ (x \in \mathbb{R}) \quad :$

$$\mathbb{R} \ni x \longrightarrow \cos(x + \pi/3); \quad \mathbb{R} \ni x \longmapsto \cos(x - \pi); \quad \mathbb{R} \ni x \longmapsto \cos(2x);$$

$$\mathbb{R} \ni x \longrightarrow 4{\cdot}\cos x; \quad \mathbb{R} \ni x \longmapsto -\frac{1}{2}{\cdot}\cos x; \quad \mathbb{R} \ni x \longmapsto \cos(-x); \quad \mathbb{R} \ni x \longrightarrow \cos x - 1;$$

$$\mathbb{R} \ni x \longrightarrow 2 + \cos x; \quad \mathbb{R} \ni x \longrightarrow |\cos x|; \quad \mathbb{R} \ni x \longrightarrow \cos(|x|).$$

6. Consider the function $f(x) := e^x \ (x \in \mathbb{R})$:

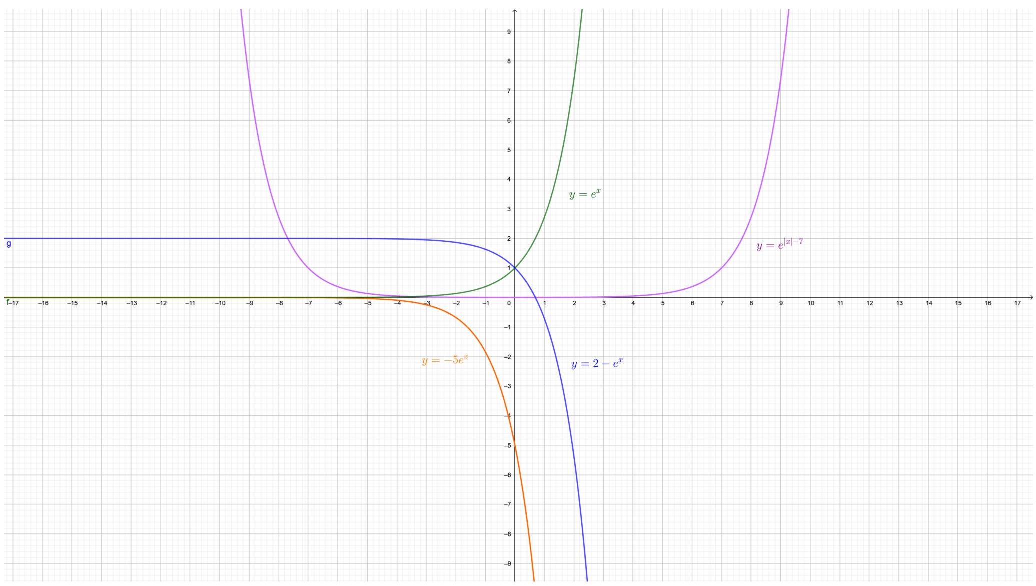
$$\mathbb{R} \ni x \longrightarrow e^{x-1}; \quad \mathbb{R} \ni x \longmapsto e^{x+2}; \quad \mathbb{R} \ni x \longmapsto e^x - 1; \quad \mathbb{R} \ni x \longmapsto e^x + 5;$$

$$\mathbb{R} \ni x \longrightarrow 2 \cdot e^x; \quad \mathbb{R} \ni x \longmapsto -e^x; \quad \mathbb{R} \ni x \longmapsto e^{-x};$$

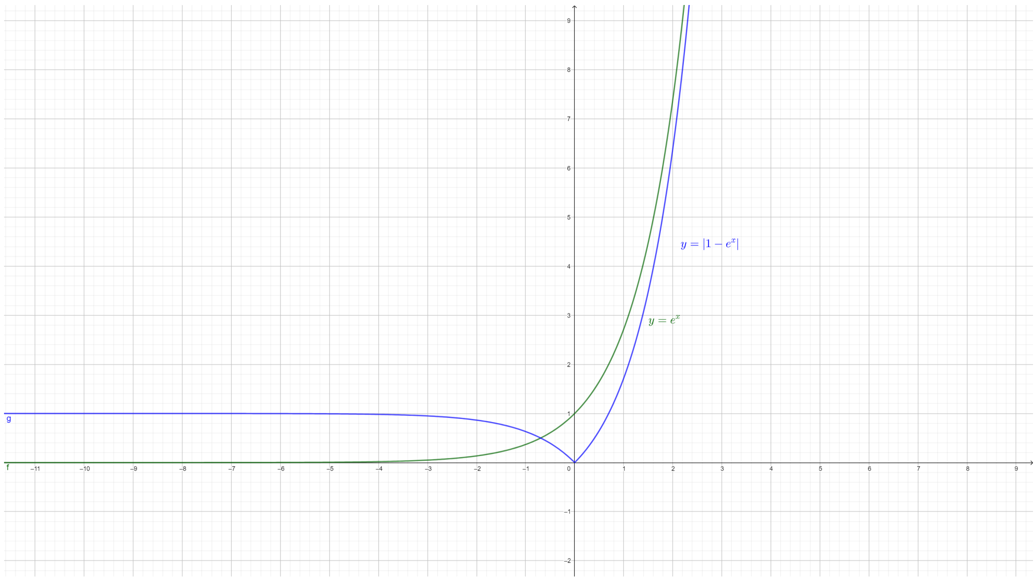
$$\mathbb{R} \ni x \longrightarrow |1 - e^x|; \quad \mathbb{R} \ni x \longrightarrow e^{|x|}.$$

See some transformations and their graphs below:

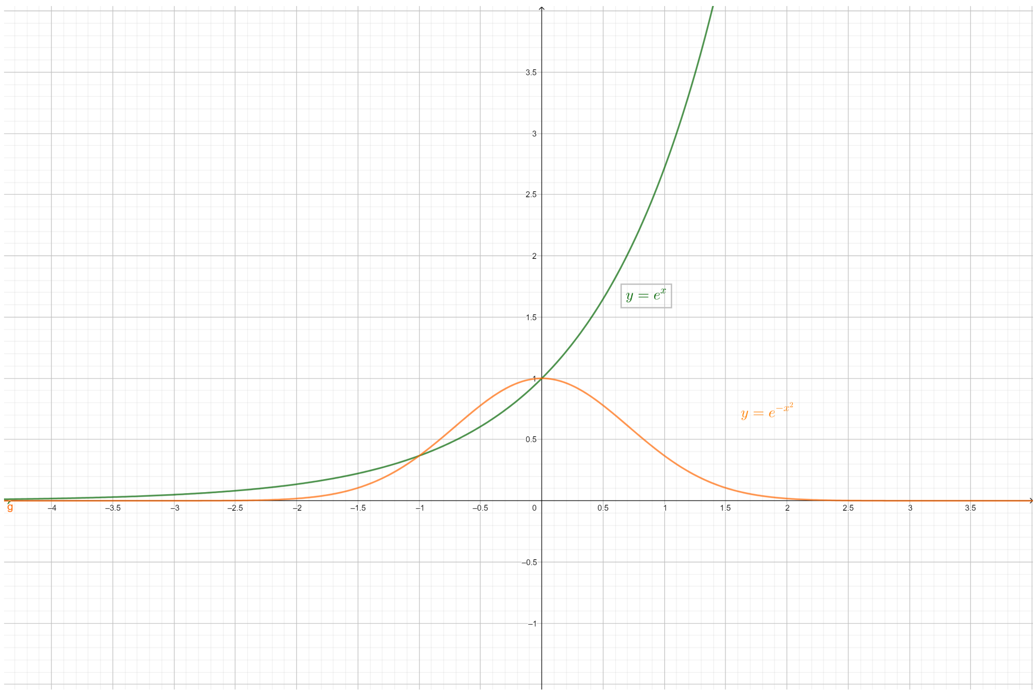
See more graphs below:



See the following transformation and graphs:

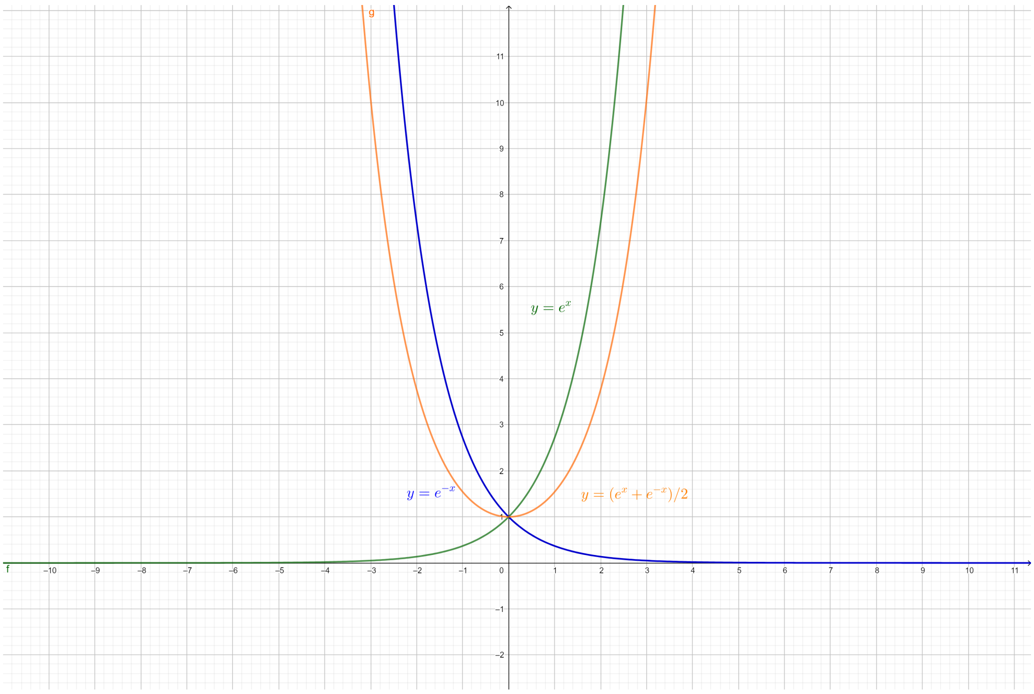


An interesting graph:

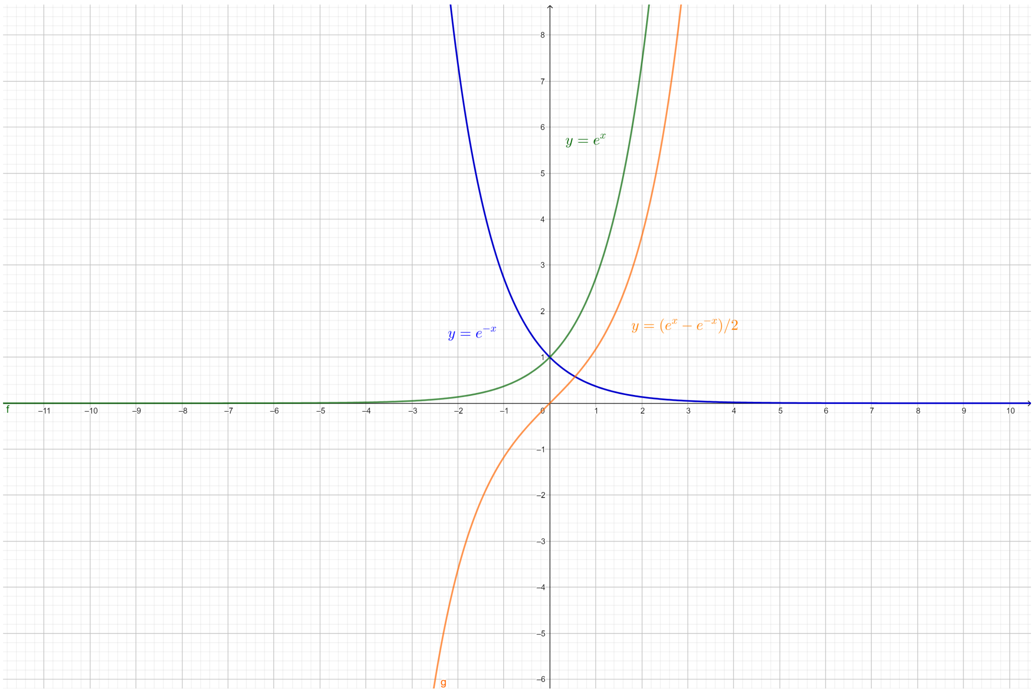


One more transformation:

Two more interesting graphs:



Here is the second one:



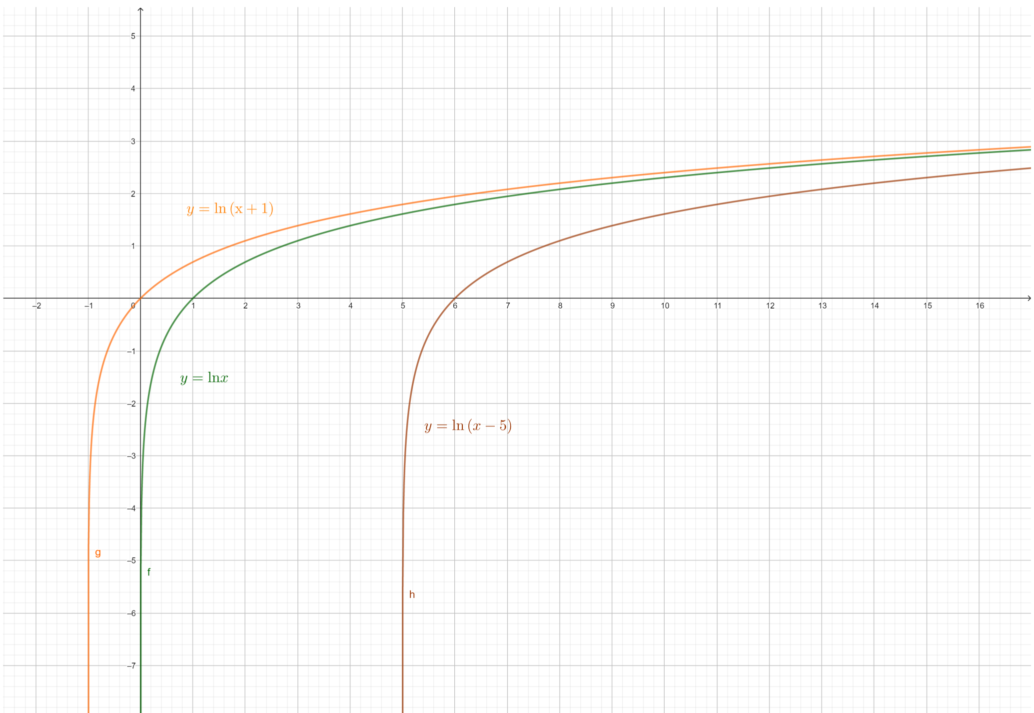
7. Consider the function $f(x) := \ln x \ \ (x \in (0; +\infty))$:

$$-1 < x \longrightarrow \ln(x+1); \quad 2 < x \longmapsto \ln(x-2); \quad 0 < x \longmapsto \ln x - 1; \quad 0 < x \longmapsto 5 + \ln x;$$

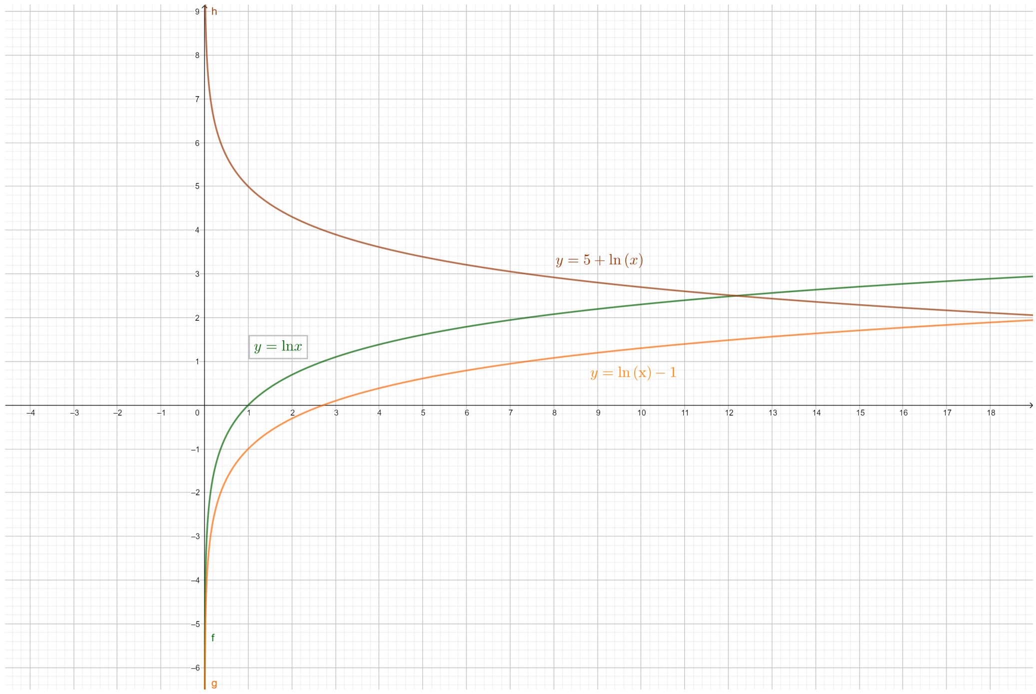
$$0 < x \longrightarrow 2 \cdot \ln x; \quad 0 < x \longrightarrow \ln(2x); \quad 0 < x \longmapsto -\ln x; \quad 0 > x \longmapsto \ln(-x);$$

$$0 < x \longrightarrow |\ln x|; \quad \mathbb{R} \setminus \{0\} \ni x \longrightarrow \ln|x|.$$

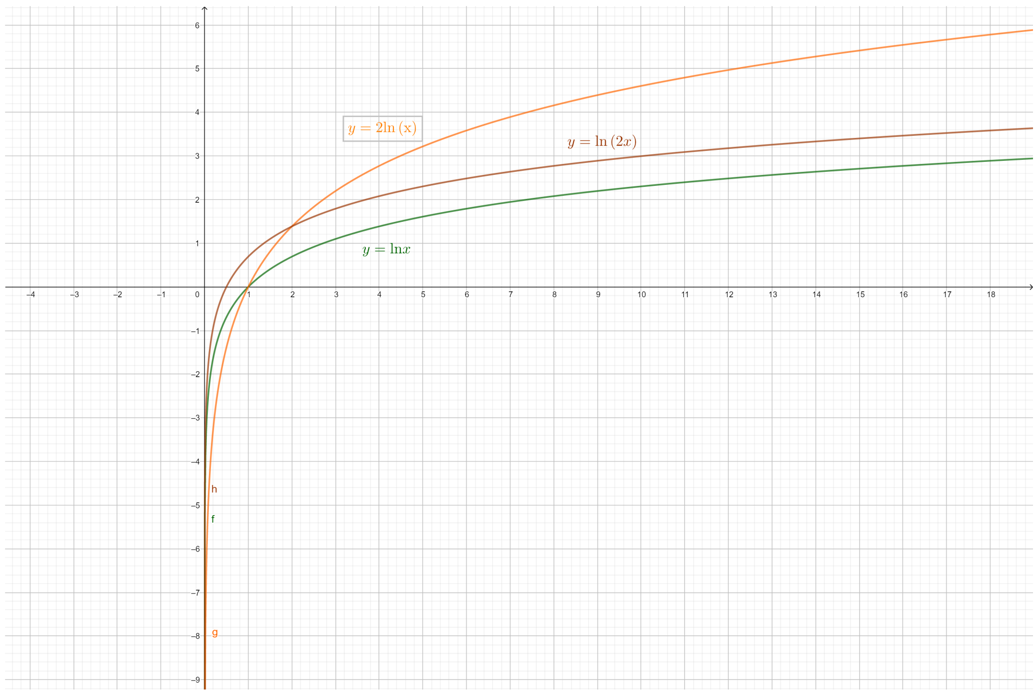
See some transformations and their graphs below:



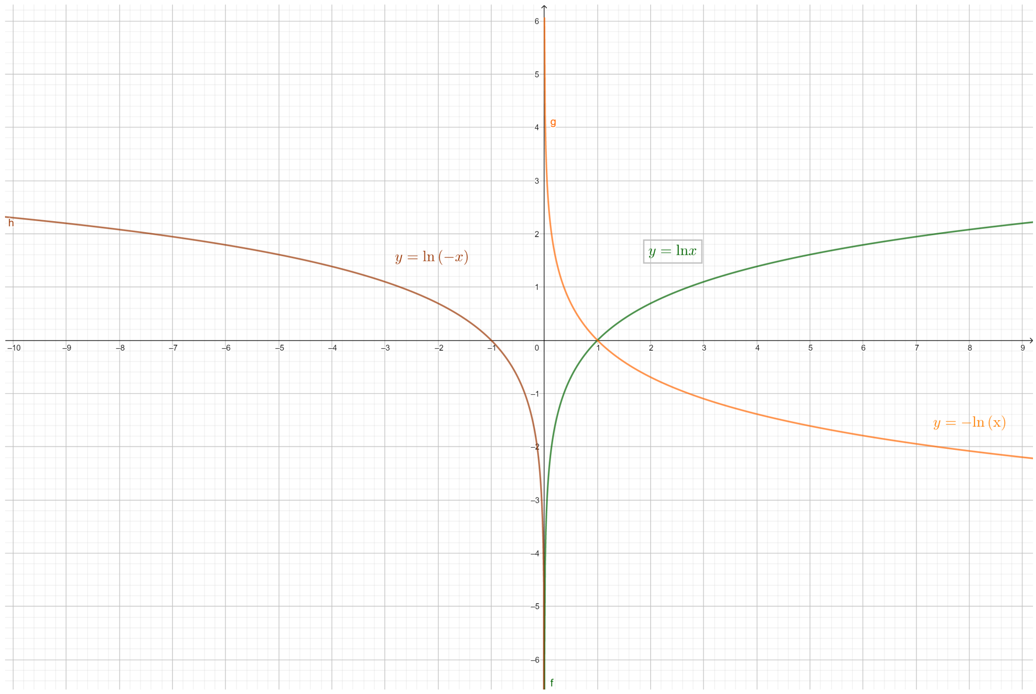
See more graphs below:



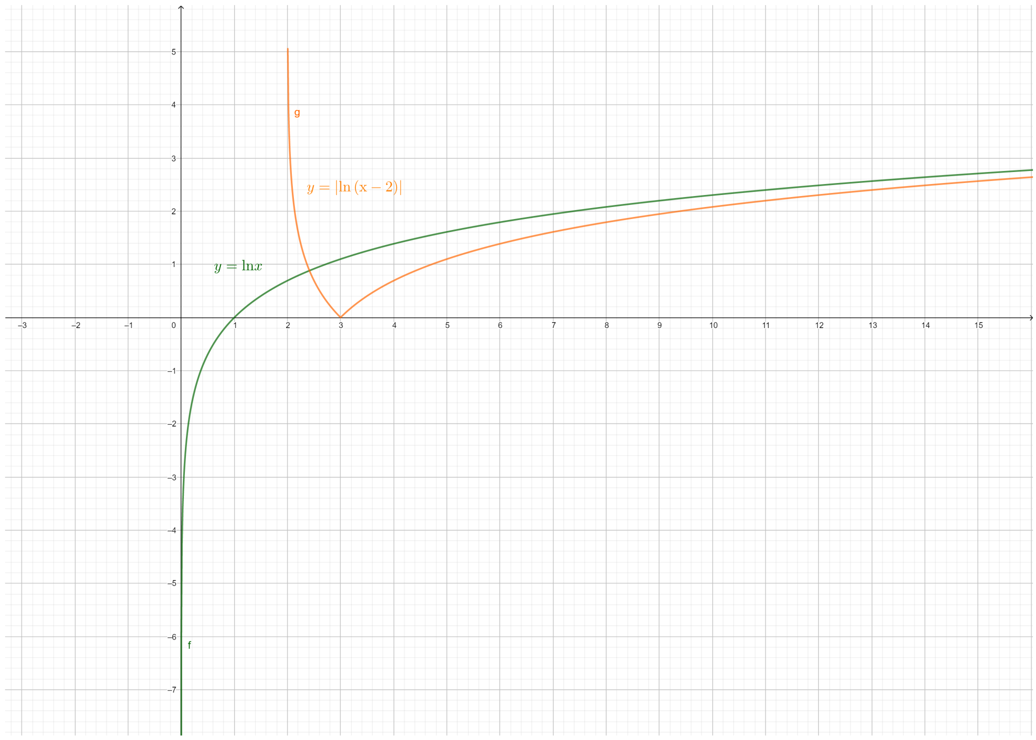
See the following transformations and graphs:



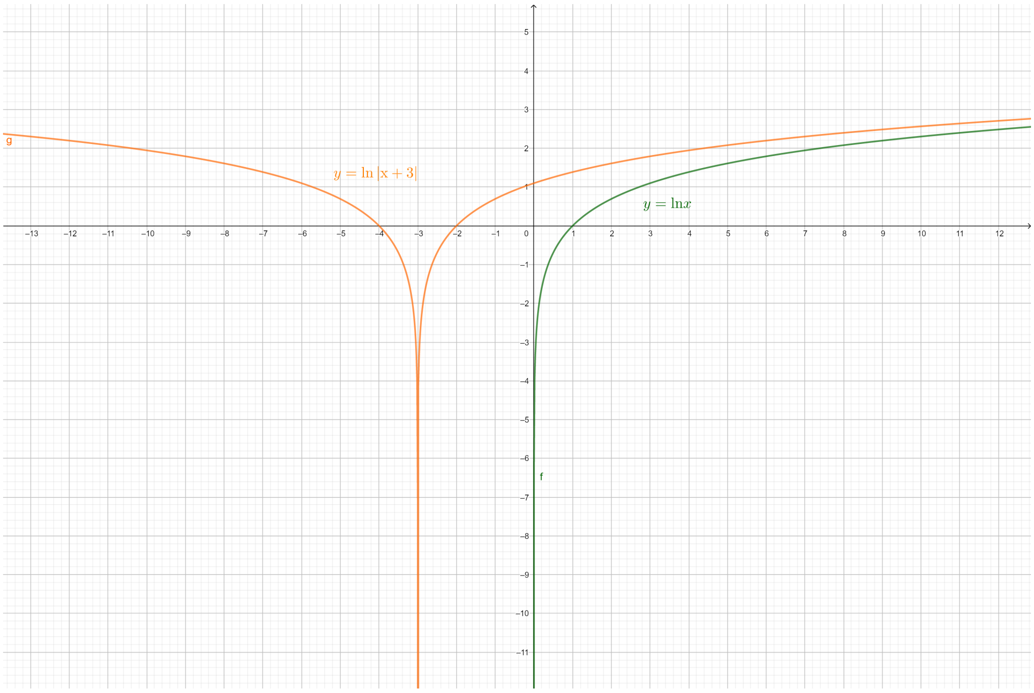
More graphs:



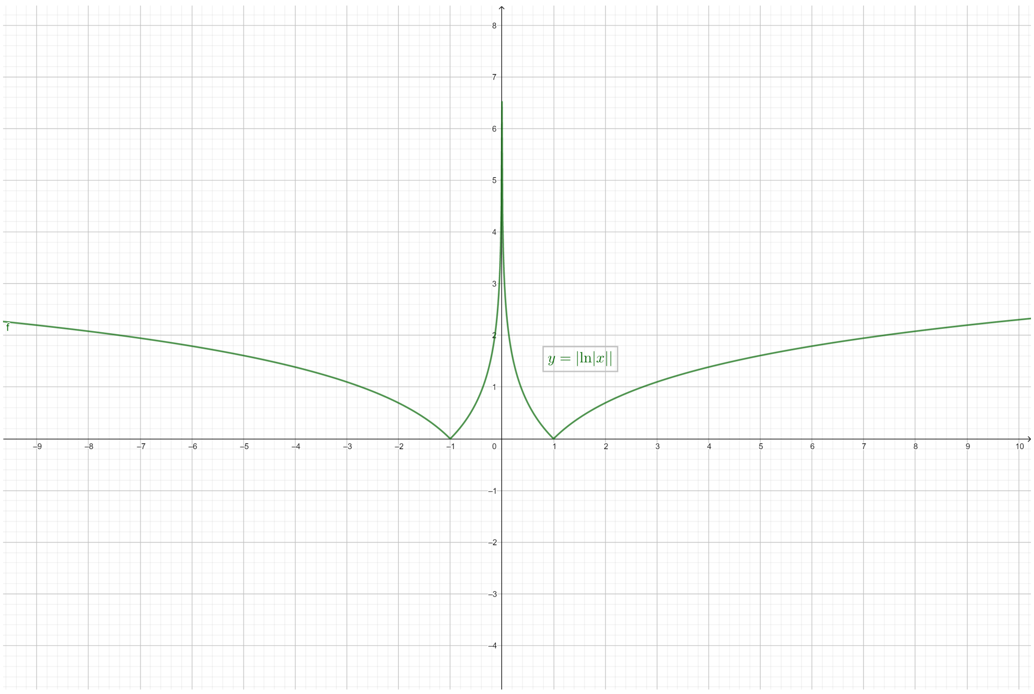
A few more graphs:



Two more interesting graphs:



Here is the second one:



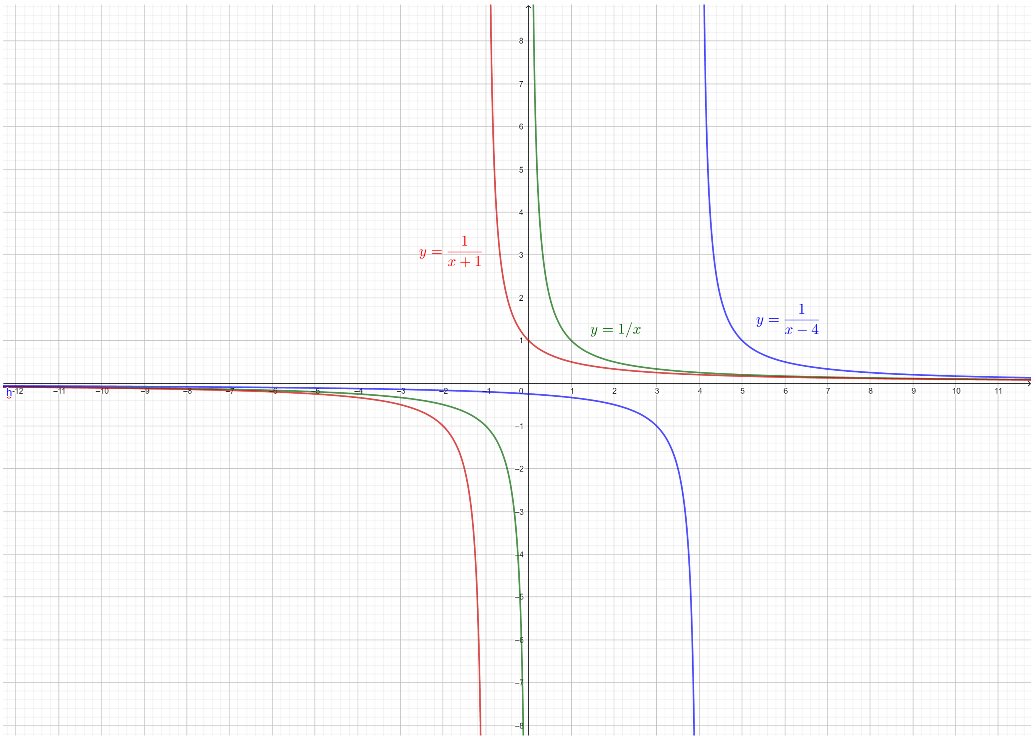
8. Consider the function $f(x) := \dfrac{1}{x}$ $(x \in \mathbb{R} \setminus \{0\})$:

$$\mathbb{R} \setminus \{-1\} \ni x \longrightarrow \frac{1}{x+1}; \quad \mathbb{R} \setminus \{1\} \ni x \longrightarrow \frac{1}{x-1}; \quad \mathbb{R} \setminus \{0\} \ni x \longrightarrow \frac{1}{x} - 1;$$

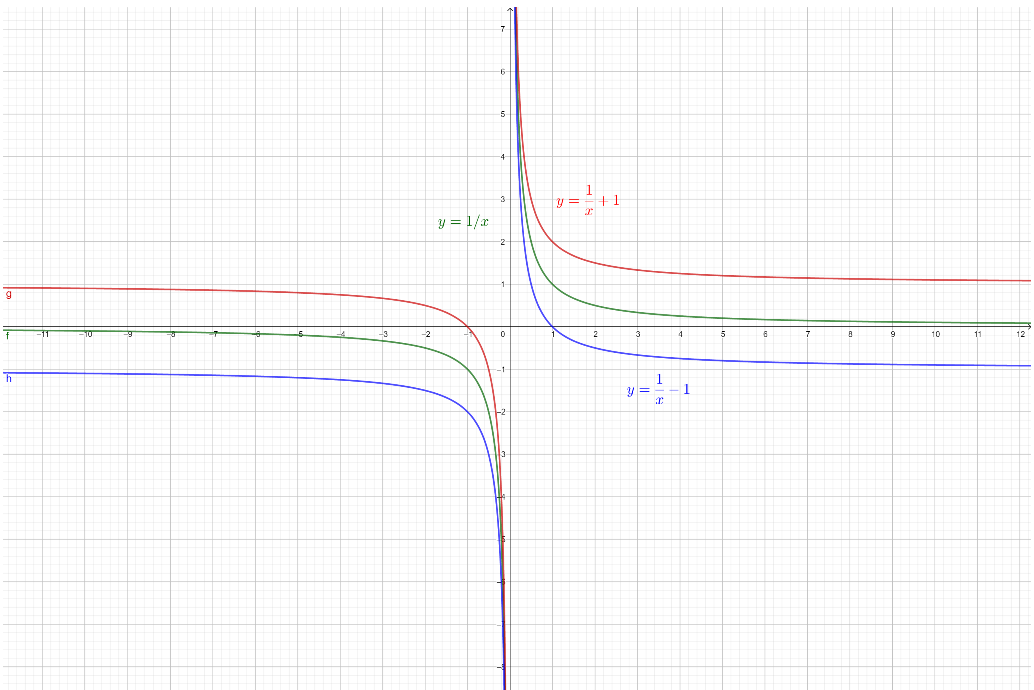
$$\mathbb{R} \setminus \{0\} \ni x \longrightarrow 1 + \frac{1}{x}; \quad \mathbb{R} \setminus \{0\} \ni x \longrightarrow \frac{2}{x}; \quad \mathbb{R} \setminus \{0\} \ni x \longrightarrow \frac{1}{2x};$$

$$\mathbb{R} \setminus \{0\} \ni x \longrightarrow \frac{1}{-x}; \quad \mathbb{R} \setminus \{0\} \ni x \longrightarrow \frac{1}{|x|}; \quad \mathbb{R} \setminus \{0\} \ni x \longrightarrow \left|\frac{1}{x}\right|.$$

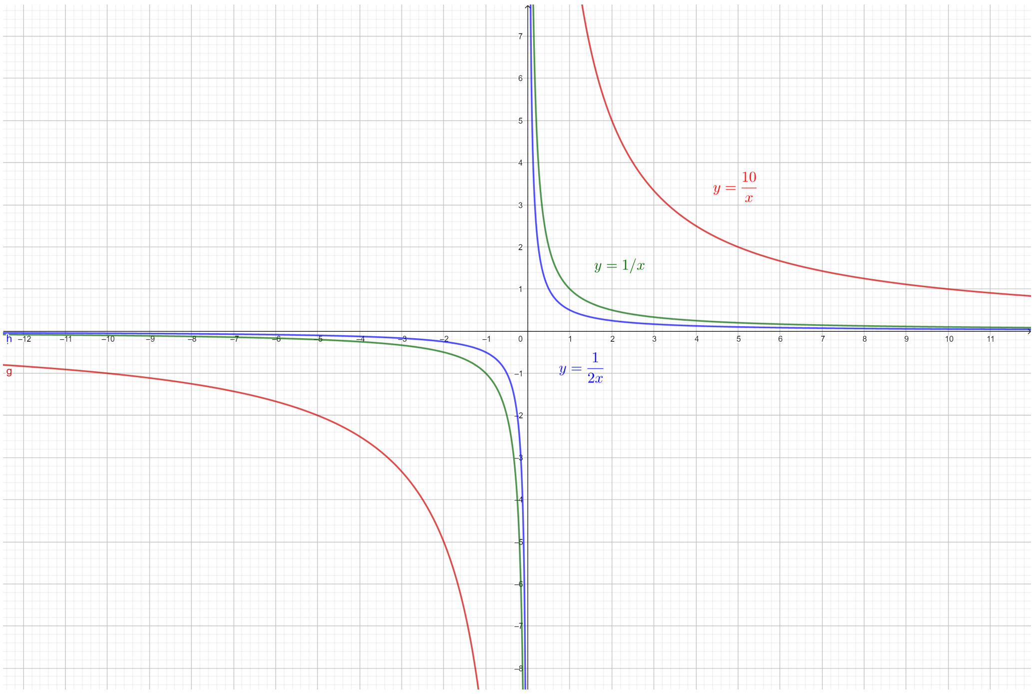
See some transformations and their graphs below:



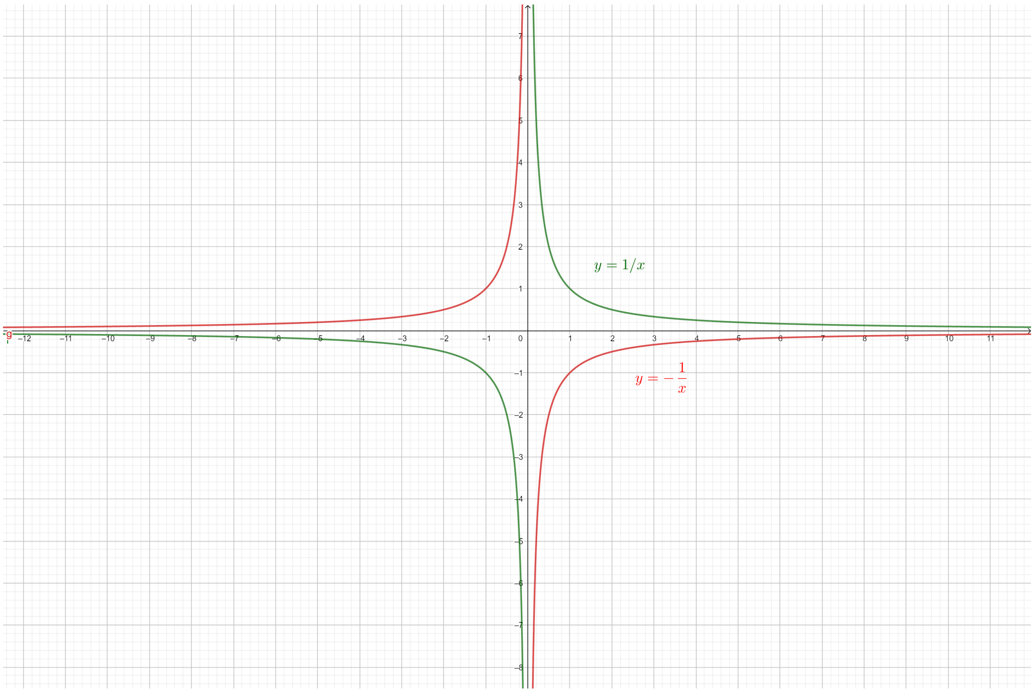
See more graphs below:



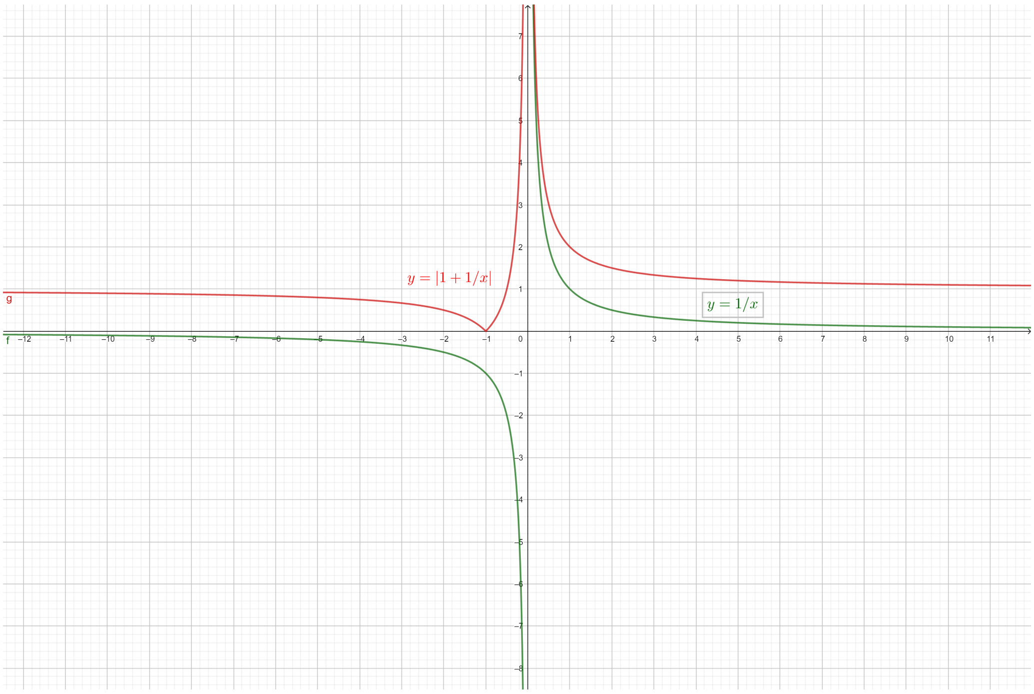
See the following transformations and graphs:



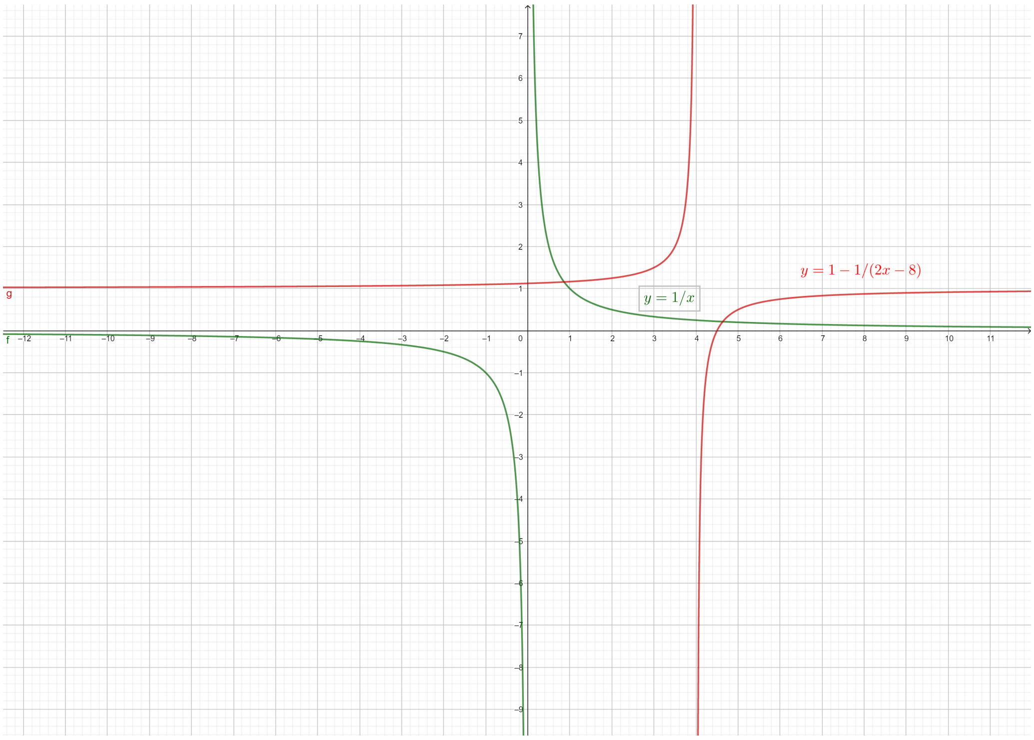
More graphs:



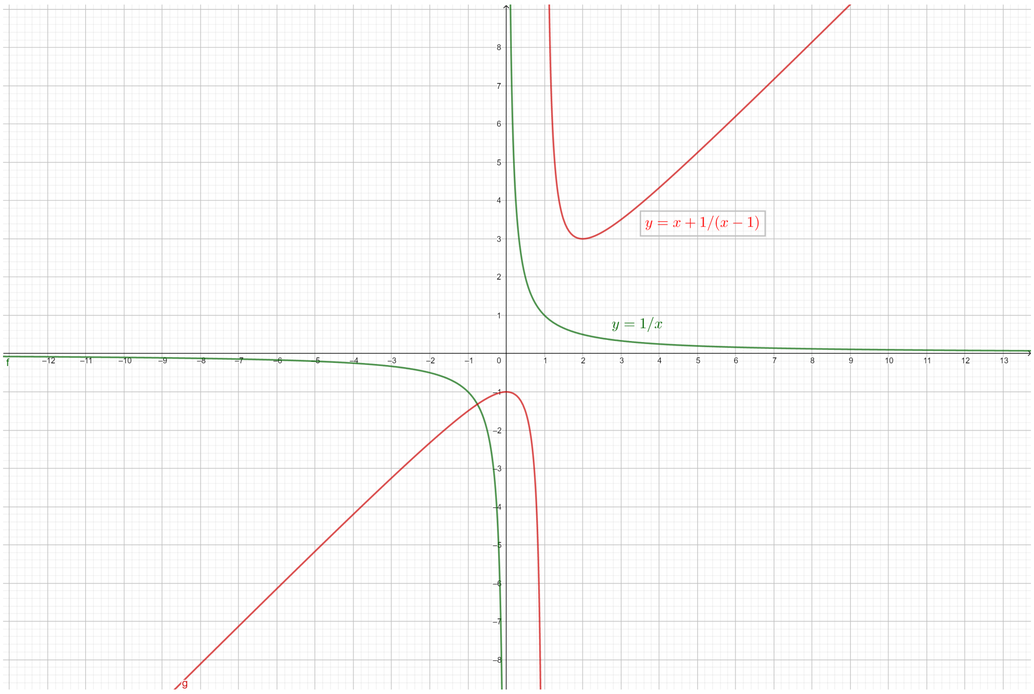
A few more graphs:



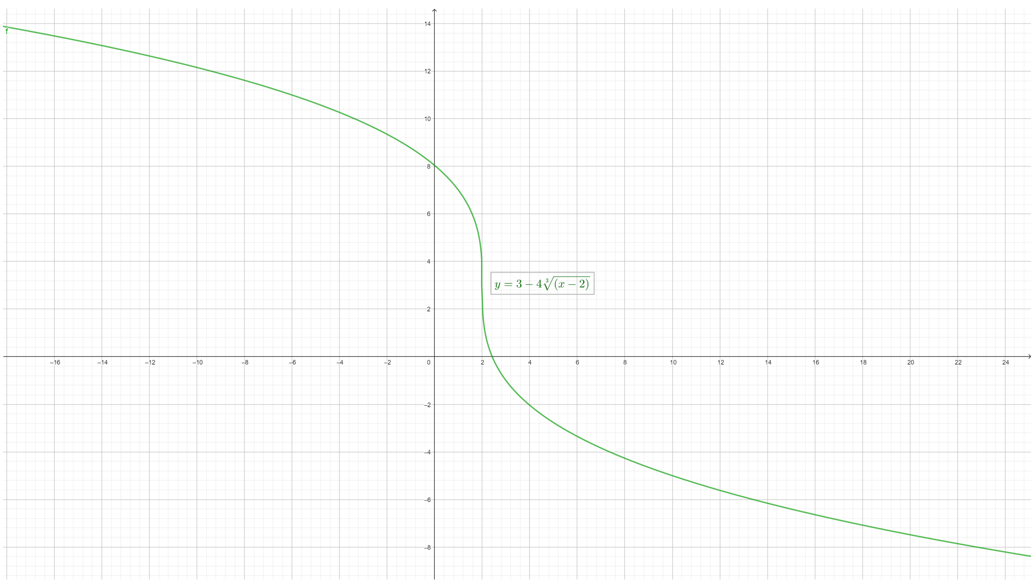
Two more interesting graphs:



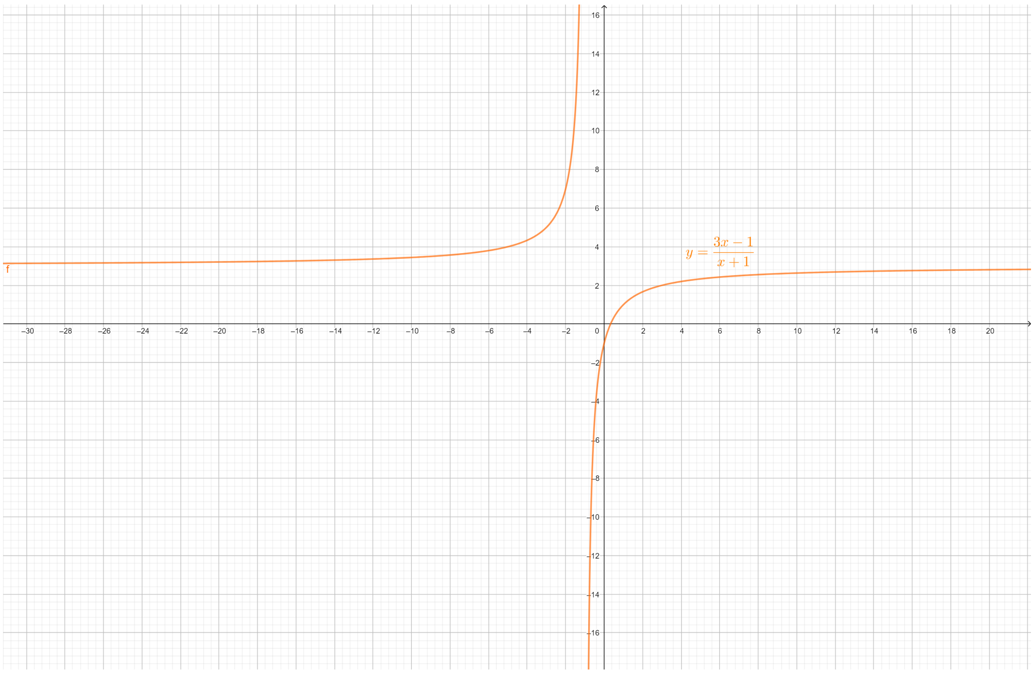
Here is the second one:



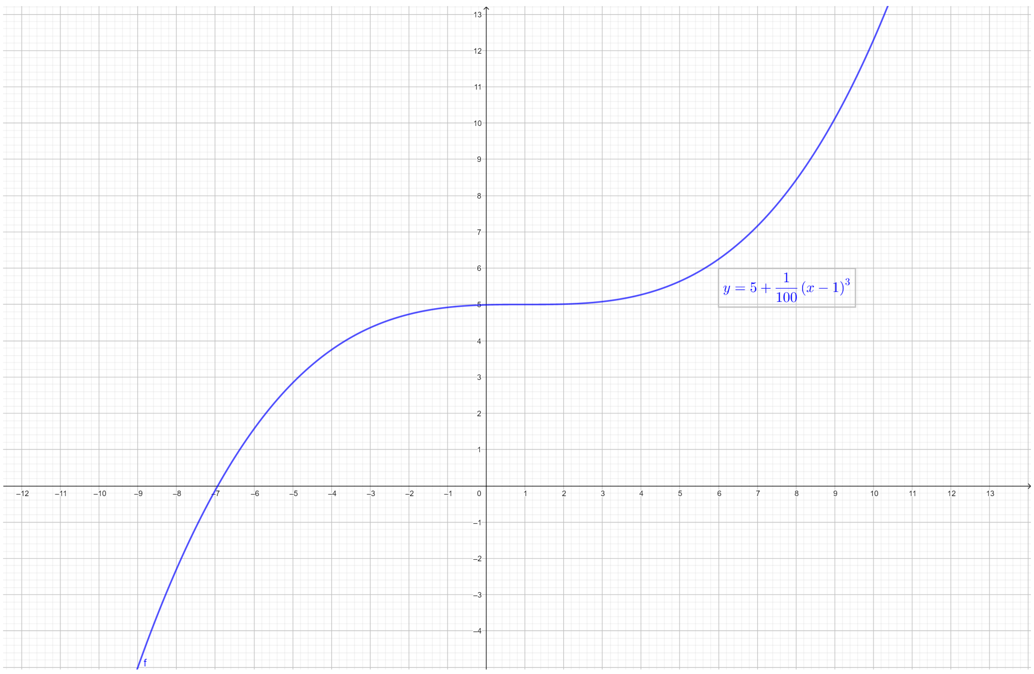
Finally here are a few graphs of some transformated functions. Try to find the step-by-step way to get the final graphs:



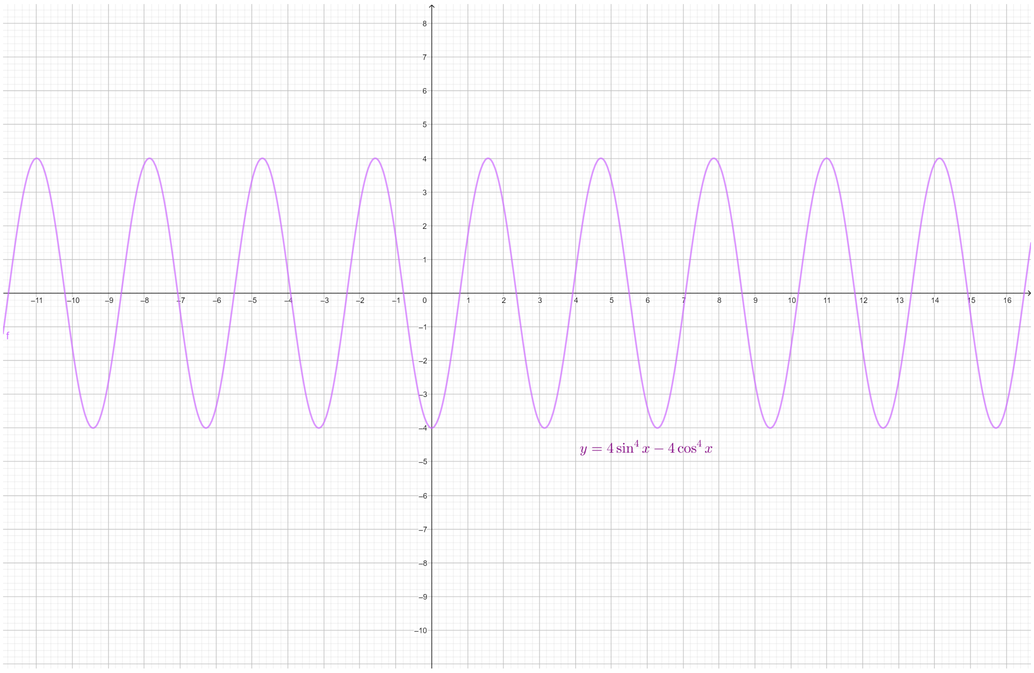
Rational fraction's graph:



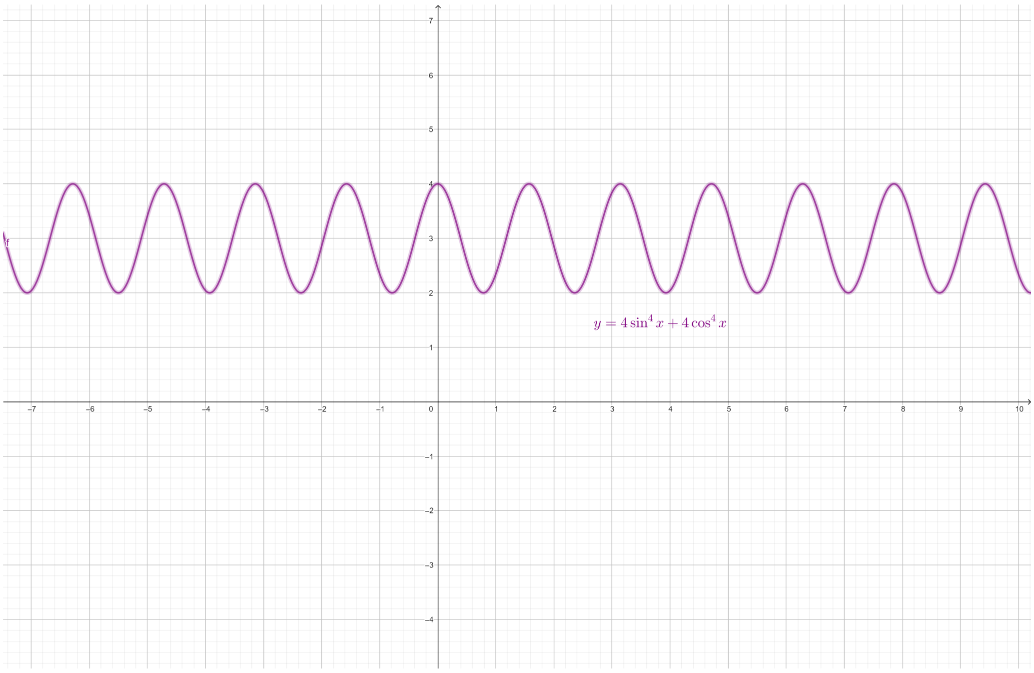
Transformated power-function's graph:



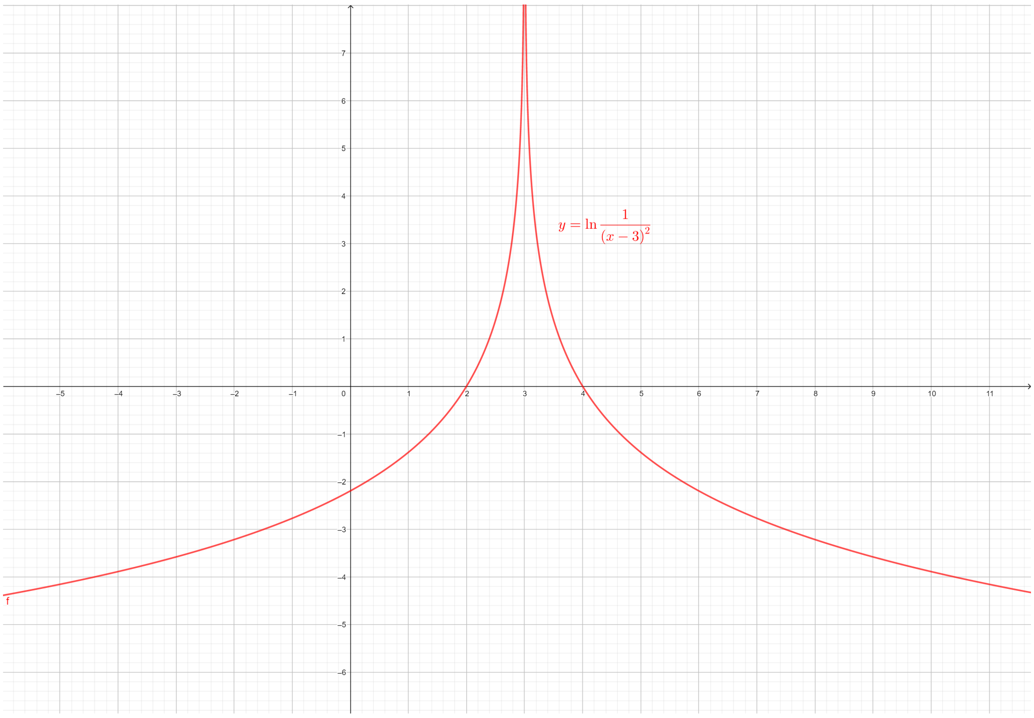
Trigonometric function:



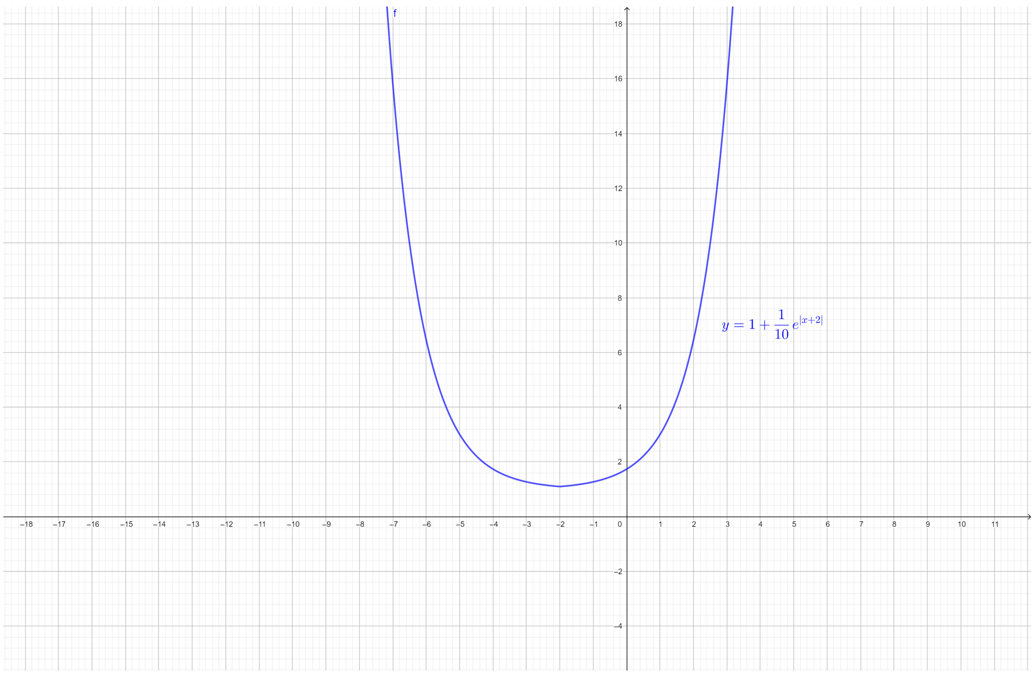
Another trigonometric function:



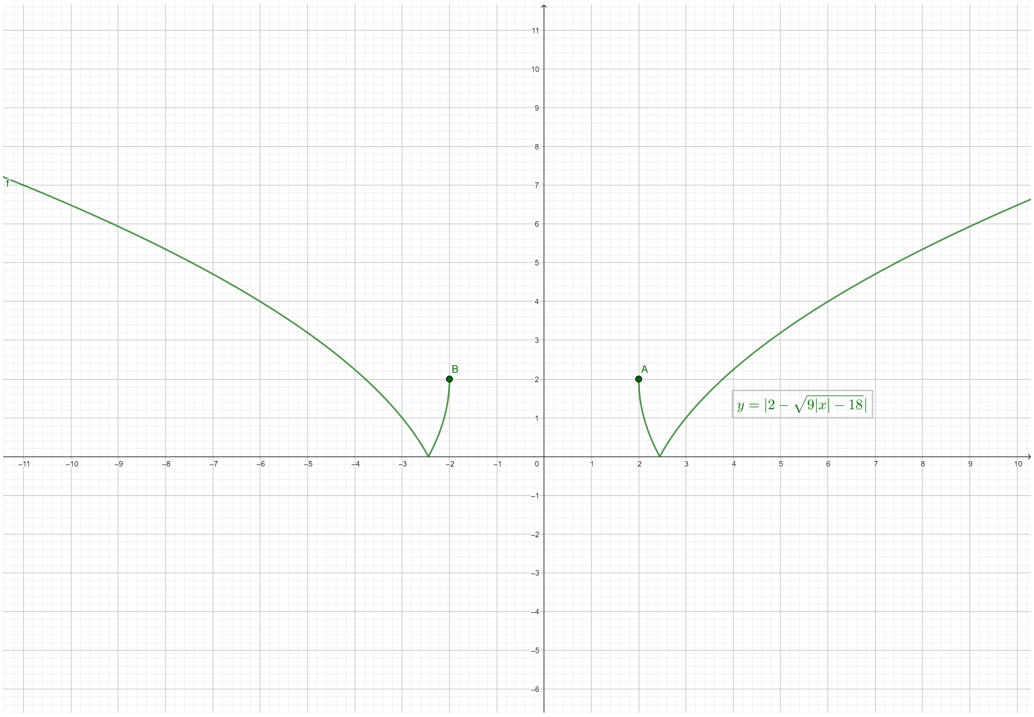
A logarithmic transformation:



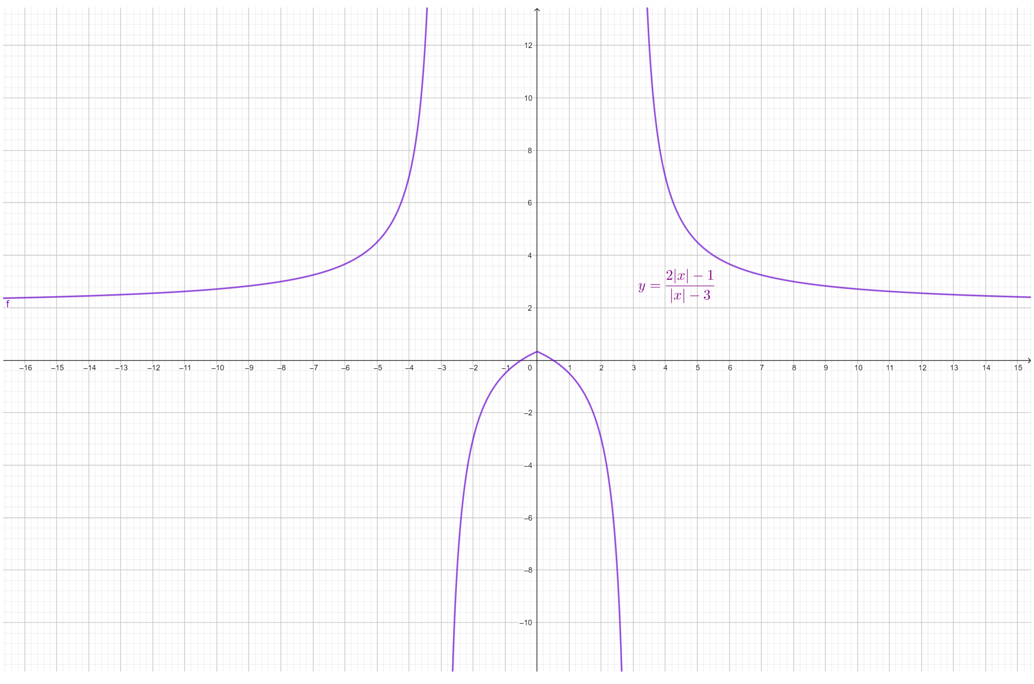
Exponential transformation and absolute value included:



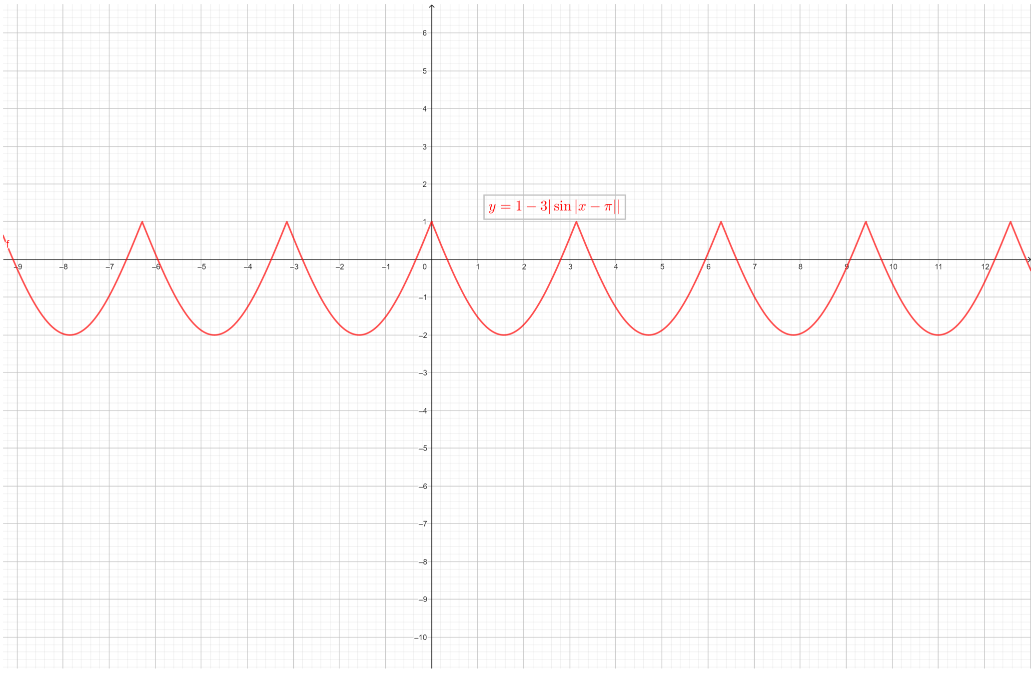
Square-root and absolute value:



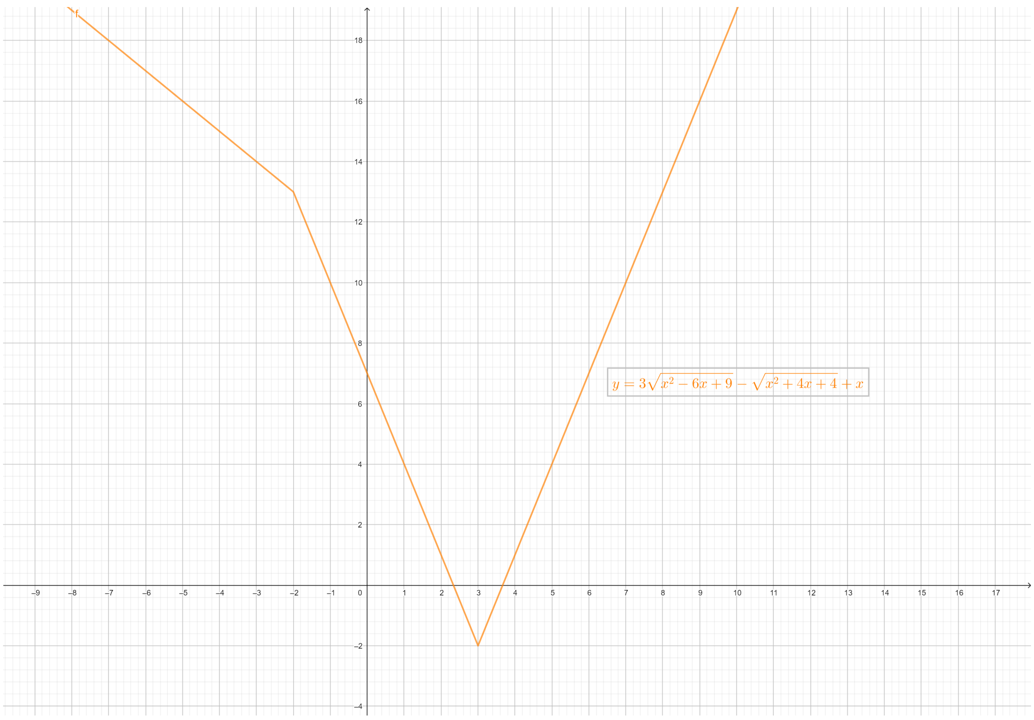
Absolute value combined with fraction:



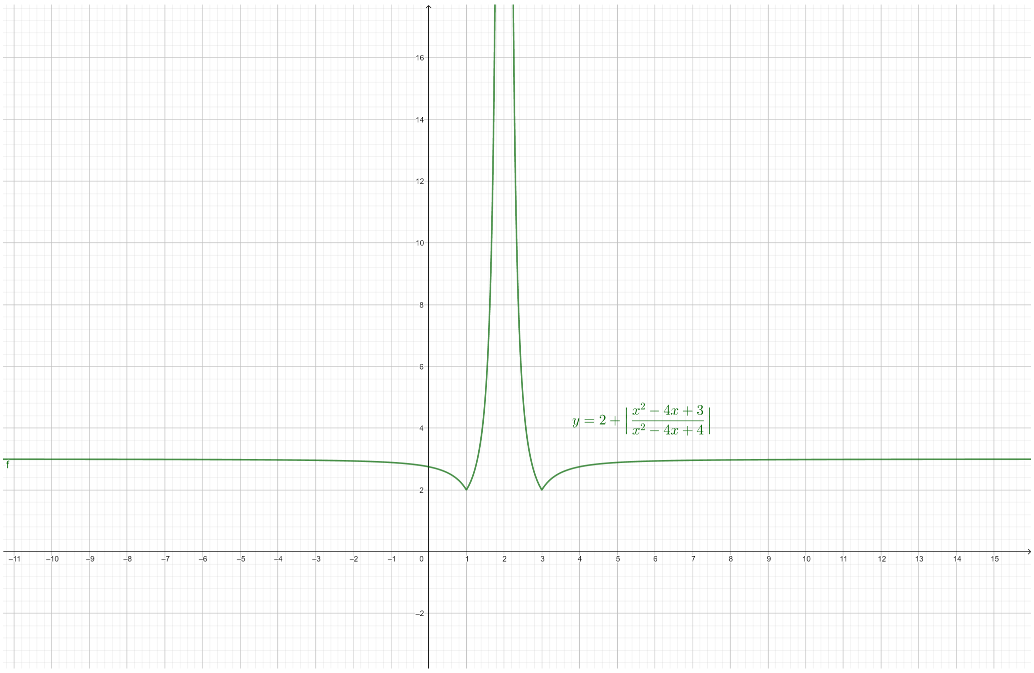
Trigonometric transformation with absolute values:



Linear combination of transformed absolute values:



Rational fraction with absolute value:



Cubic function with absolute value:

1.1.1. Checking questions to the theory and its use

1. Define the notion of a *function.*

2. Define the *quotient function* g/f.

3. Define the *composition function* $g \circ f$.

4. Define the *graph* of a function $f : A \longrightarrow B$.

5. Define the *range of values* R_f of a function $f \in A \longrightarrow B$.

6. Illustrate in the plane all the points of the set $(1; 2] \times [-1; 1)$.

7. Give a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ for which the function $f^2 := f \cdot f$ is equal to f. Give another one as well, when $f^2 \neq f$.

8. Suppose, that the product of the functions f and g is the constant 0 function, so:
$$f \cdot g : \mathbb{R} \longrightarrow \mathbb{R} \ \ (f \cdot g)(x) = 0 \quad (x \in \mathbb{R}).$$
Is it true, that f or g is the constant 0 function?

9. Consider the functions $f(x) := \sqrt{x-1}$ $(x \in [1; +\infty))$ and $g(x) := x - 1$ $(x \in \mathbb{R})$. Is it true, that $f^2 = g$? What restriction of g will be equal to f^2?

10. Give the function $f \circ f$ if $f(x) := 3^x$ $(x \in \mathbb{R})$. What is $(f \circ f)(0)$ and $(f \circ f)(-1)$?

11. Give the functions $f \pm g$, $f \cdot g$, f/g and g/f if:
$$f(x) := \frac{1}{x^2 - 1} \ \ (x \in \mathbb{R} \setminus \{-1; 1\}) \ \ \wedge \ \ g(x) := \sqrt{e^x - 1} \ \ (x \in [0; +\infty)).$$

12. Give the functions $f \circ g$ and $g \circ f$ if:
$$f(x) := \sqrt{x} \ \ (x \in (0; +\infty)) \ \ \wedge \ \ g(x) := x^2 \ \ (x \in \mathbb{R}).$$

13. Give the largest set $D \subseteq \mathbb{R}$ so, that f defines a function:
$$f(x) := \frac{\ln(3 - \sqrt{x})}{\sqrt{1 - \sin x}} \ \ (x \in D).$$

14. Sketch the graph of the function:
$$f(x) := 2^{|x|} \ \ (x \in \mathbb{R}).$$

15. Sketch the graph of the function:
$$f(x) := \frac{x - 1}{\sqrt{x} - 1} \quad (1 \neq x \in (0; +\infty)).$$

16. Define the notion of the *ordered pair* $(a; b)$.

17. Define the domain of definition D_f of a function $\emptyset \neq f \subseteq A \times B$.

18. Find $A \times B$ if $A := \{1\}$ and $B := [0; 3]$. Illustrate all the points of $A \times B$ in the Cartesian coordinate system.

19. Give $A \times B$ when $A := (-1; 1]$ and $B := \{2\}$. Draw the points of this set $A \times B$ in $\mathbb{R}^2$.

20. What does the notation $f \in A \longrightarrow B$ mean?

21. What do we mean by the notation $f : A \longrightarrow B$?

22. Give the definition of a *relation*.

23. When do we say, that the functions f and g are *equal* to each other?

24. Give the functions $f \circ g$ and $g \circ f$ if:

$$f(x) := \ln(-2 - x) \;\; (x \in (-\infty; -2)) \;\; \wedge \;\; g(x) := \cos x \;\; (x \in \mathbb{R}).$$

25. What is the definition of the *Cartesian / Descartes product* of two sets?

26. Define the *square-root* and the *cube-root* functions and draw their graphs as well.

27. Draw the graph of the *identity* function ($f(x) := x \;\; (x \in \mathbb{R})$).

28. Give the graph of the function:

$$f(x) := 2 - 3x \;\; (x \in \mathbb{R}).$$

29. Give the graph of the function:

$$f(x) := 2 - x^2 \;\; (x \in \mathbb{R}).$$

30. Define the *sum-function* $f + g$.

31. What is the definition of the *difference-function* $g - f$.

32. Define the *product-function* $f \cdot g$ of two functions f and g.

33. Give the definition and the graph of the *absolute-value* function.

34. Give the definition and the graph of the *sign* function.

35. Give the definition and the graph of the *integer-part* function.

36. Give the definition and the graph of the *constant* 5 function.

37. Give the graph of the *reciprocal* function

$$f(x) := \frac{1}{x} \ \ (x \in \mathbb{R} \setminus \{0\}).$$

38. Give the graph of the function

$$f(x) := \frac{1}{x^2} \ \ (x \in \mathbb{R} \setminus \{0\}).$$

39. Give the graph of the cube-function

$$f(x) := x^3 \ \ (x \in \mathbb{R}).$$

40. Sketch the graphs of

$$f(x) := \sin x \ \ (x \in \mathbb{R}) \ \wedge \ g(x) = \cos x \ \ \ (x \in [0; 3\pi]).$$

41. Sketch the graphs of:

$$f(x) := \mathrm{tg}x \ \ (x \in (-\pi/2; \pi/2)) \ \ \wedge \ \ g(x) = \mathrm{ctg}x \ \ \ (x \in \mathbb{R} \setminus \{k\pi \ \Big| \ k \in \mathbb{Z}\}).$$

42. Sketch the graphs of the following functions in the same coordinate system:

$$f(x) := e^x \ \ (x \in \mathbb{R}) \ \ \wedge \ \ g(x) = \ln{(x)} \ \ \ (x \in (0; +\infty)).$$

43. Sketch the graphs of the following functions in the same coordinate system:

$$f(x) := 2^{-x} \ \ (x \in \mathbb{R}) \ \ \wedge \ \ g(x) = \log_{1/2}{(x)} \ \ \ (x \in (0; +\infty)).$$

44. Draw the graph of the function:

$$f(x) := 1 + (x - 3)^3 \ \ (x \in \mathbb{R}).$$

45. Draw the graph of the function:

$$f(x) := |1 - x^2| \ \ (x \in \mathbb{R}).$$

46. Draw the graph of the function:

$$f(x) := g(x + 1) \ \ (x > -1), \text{ where } g(x) = \log_2(x) \ \ \ (x > 0).$$

47. Draw the graph of the function:

$$f(x) := g(x - 4) \ \ (x \in \mathbb{R}), \text{ where } g(x) = \sqrt[3]{x} \ \ \ (x \in \mathbb{R}).$$

48. Draw the graph of the function:

$$f(x) := |g(-x)| \ \ (x \in \mathbb{R}), \text{ where } g(x) = 2x + 1 \ \ \ (x \in \mathbb{R}).$$

49. Draw the graph of the function:

$$f(x) := -1 + 2 \cdot \cos(x - \pi) \quad (x \in \mathbb{R}).$$

50. Draw the graph of the function:

$$f(x) := \left|2 - \frac{1}{x}\right| \quad (x \in \mathbb{R} \setminus \{0\}).$$

51. Draw the graph of the function:

$$f(x) := \left|2 - \sqrt{1-x}\right| \quad (x \in (-\infty; 1]).$$

1.2. Exercises

1.2.1. Exercises for class work

Domain of definition

1. Give the largest set $D \subset \mathbb{R}$ for which the following expressions define a real-real funcion f:

   (a) $f(x) := \sqrt{\dfrac{2x^3 - 1}{x}} \quad (x \in D)\,;$

   (b) $f(x) := \sqrt{\lg\left(x^2 - 5x + 7\right)} \quad (x \in D)\,;$

   (c) $f(x) := \sqrt{2^x - e^x} + \sqrt{3^x - e^x} \quad (x \in D)\,;$

   (d) $f(x) := \dfrac{\sqrt{16 - x^2}}{\lg(\sin x)} \quad (x \in D)\,.$

Equality of functions

2. Are the following functions equal or not:

   (a) $f(x) := \sqrt{x} \;\; (x \in [0; +\infty));\;\; g(x) := \sqrt[4]{x^2} \;\; (x \in \mathbb{R})$?

   (b) $f(x) := \sqrt{x} \;\; (x \in [0; +\infty));\;\; g(x) := \sqrt[4]{x^2} \;\; (x \in [0; +\infty))$?

   (c) $f(x) := \sqrt{x^2} \;\; (x \in \mathbb{R});\;\; g(x) := |x| \;\; (x \in \mathbb{R})$?

(d) $f(x) := \sqrt{x^2}$ $(x \in \mathbb{R})$; $g(x) := (\sqrt{x})^2$ $(x \in [0;+\infty))$?

Can we say, that $f|_{[0;+\infty)} = g$?

(e) $f(x) := \ln(x^2)$ $(x \in \mathbb{R} \setminus \{0\})$; $g(x) := 2 \cdot \ln x$ $(x \in (0;+\infty))$?

(f) $f(x) := \ln(x^2)$ $(x > 0)$; $g(x) := 2 \cdot \ln x$ $(x \in (0;+\infty))$?

(g) $f(x) := \dfrac{x}{|x|}$ $(x \in \mathbb{R}\setminus\{0\}) \wedge f(0) := 0$; $g(x) := \dfrac{|x|}{x}$ $(x \in \mathbb{R}\setminus\{0\}) \wedge g(0) := 0$?

(h) $f(x) := e^{\ln x}$ $(x \in \mathbb{R}^+)$; $g(x) := \ln(e^x)$ $(x \in \mathbb{R})$?

(i) $f(x) := \dfrac{1-\cos x}{2}$ $(x \in [0;\pi/2])$; $g(x) := \sin^2 \dfrac{x}{2}$ $(x \in [0;\pi])$?

Is it true, that $f = g|_{[0;\pi/2]}$?

Sketch the graphs of these functions.

Transformations and graphs

3. Draw the graphs of the following *special* functions and using this graph give their *range of values* R_f as well:

(a)

$$f(x) := \mathrm{Abs}(x) := |x| = \begin{cases} -x, & \text{if } x < 0; \\ x, & \text{if } x \geq 0. \end{cases}$$

(b) $f(x) := [x]$ $(x \in \mathbb{R})$; where $[x] :=$ the greatest integer number not greater then x. (Integer part function);

(c) $f(x) := \{x\} := x - [x]$ $(x \in \mathbb{R})$; (Fractional part function);

(d) Sign or Signum function:

$$f(x) := \mathrm{Sign}(x) := \begin{cases} -1, & \text{if } x < 0; \\ 0, & \text{if } x = 0; \\ 1, & \text{if } x > 0. \end{cases}$$

(e) Dirichlet function:

$$D(x) := \begin{cases} 1, & \text{if } x \in \mathbb{R} \cap \mathbb{Q}; \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

(f) Riemann function:

$$R(x) := \begin{cases} \dfrac{1}{q}, & \text{if } x = \dfrac{p}{q} \in \mathbb{Q},\ p, q \in \mathbb{Z}\ \wedge\ (p;q) = 1\ \wedge\ q > 0; \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

4. Give the graphs of the following functions. Write down the transformational steps as well.

(a) $f(x) := 2(x+3)^2 - 1 \quad (x \in \mathbb{R});$

(b) $f(x) := -x^2 + 5x + 3 \quad (x \in \mathbb{R});$

(c) $f(x) := x^3 - 3x^2 + 3x + 1 \quad (x \in \mathbb{R});$

(d) $f(x) := |x^2 - 5x + 6| \quad (x \in \mathbb{R});$

(e) $f(x) := |2 - |x-1|| \quad (x \in \mathbb{R});$

(f) $f(x) := \dfrac{x+3}{x+5} \quad (-5 \neq x \in \mathbb{R});$

(g) $f(x) := \dfrac{4x-1}{2x-1} \quad \left(\dfrac{1}{2} \neq x \in \mathbb{R}\right);$

(h) $f(x) := \dfrac{x}{1+|x|} \quad (x \in \mathbb{R});$

(i) $f(x) := \dfrac{\sqrt{5x-1}}{4} + 2 \quad \left(\dfrac{1}{5} \leq x \in \mathbb{R}\right);$

(j) $f(x) := 2 - \sqrt{1-x} \quad (x \in (-\infty; 1]);$

(k) $f(x) := \sqrt{|x|} \; (x \in \mathbb{R});$

(l) $f(x) := \sin\left(x - \dfrac{\pi}{4}\right) + \sin\left(x + \dfrac{\pi}{4}\right) \quad (x \in \mathbb{R});$

(m) $f(x) := \sin x - \sqrt{3} \cdot \cos x \quad (x \in \mathbb{R})\;;$

(n) $f(x) := \cos^2 x \quad (x \in \mathbb{R})\;;$

(o) $f(x) := \operatorname{tg}\left(\dfrac{\pi}{4} - x\right) \quad (x \in (-\pi/4; 3\pi/4))\;;$

(p) $f(x) := 3 \cdot 2^{3x-1} \quad (x \in \mathbb{R})\;;$

(q) $f(x) := e^{-x} \quad (x \in \mathbb{R})\;;$

(r) $f(x) := \ln(1-x) \quad (x \in (-\infty; 1))\;;$

(s) $f(x) := \ln\dfrac{e}{x} \quad (x \in (0; +\infty))\;;$

(t)

$$D(x) := \begin{cases} x^2 - 2x + 1, & \text{if } x \in \mathbb{R} \cap \mathbb{Q}; \\ x^3 - 3x^2 + 3x - 1, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Operations with functions

5. Consider the functions f and g. Find the given function h and where is possible draw the graphs of f, g and h:

(a) $f(x) := x \;\; (x \in \mathbb{R});\;\; g(x) := \sin x \;\; (x \in \mathbb{R});$

Find $h := f + g;\; h := f - g;\; h := f \cdot g;\; h := \dfrac{f}{g};\; h := \dfrac{g}{f}.$

(b) $f(x) := D(x)\ \ (x \in \mathbb{R});\ \ g(x) := 1 - D(x)\ \ (x \in \mathbb{R});$

Find $h := f + g;\ h := f - g;\ h := f \cdot g;\ h := \dfrac{f}{g};\ h := \dfrac{g}{f}.$

Here D is the Dirichlet function.

(c) $f(x) := |x|\ \ (x \in \mathbb{R});\ \ g(x) := x\ \ (x \in \mathbb{R});$

Find $h := f + g;\ h := f - g;\ h := f \cdot g;\ h := \dfrac{f}{g}.$

(d) $f(x) := \sin x\ \ (x \in \mathbb{R});\ \ g(x) := \cos x\ \ (x \in \mathbb{R});$

Find $h := f + g;\ h := f - g;\ h := f \cdot g;\ h := \dfrac{f}{g};\ h := \dfrac{g}{f}.$

(e) $f(x) := \sqrt{x} - 1\ \ (x \in [0; +\infty));\ \ g(x) := 1\ \ (x \in \mathbb{R});$

Find $h := f + g;\ h := f - g;\ h := f \cdot g;\ h := \dfrac{f}{g};\ h := \dfrac{g}{f}.$

(f) $f(x) := x\ \ (x \in \mathbb{R});\ \ g(x) := [x]\ \ (x \in \mathbb{R});$

Find $h := f + g;\ h := f - g;\ h := f \cdot g;\ h := \dfrac{f}{g};\ h := \dfrac{g}{f}.$

6. Consider the following functions:

$$f(x) := \frac{1}{2} \cdot e^x\ \ (x \in \mathbb{R});\ \ g(x) := \frac{1}{2} \cdot e^{-x}\ \ (x \in \mathbb{R});$$

i) Find the following functions:

(a) $\mathrm{ch} := f + g;\ \mathrm{sh} := f - g;\ p := f \cdot g;\ l := \dfrac{f}{g};\ t := \dfrac{g}{f}.$

(b) $w := \mathrm{ch}^2 - \mathrm{sh}^2;\ \ v := 2 \cdot \mathrm{ch} \cdot \mathrm{sh};\ \ r := \mathrm{ch}^2 + \mathrm{sh}^2.$

ii) Prove, that for all $y \in [1; +\infty)$ the equation

$$\mathrm{ch}(x) = y$$

can be solved (related to x) on the set of positive real numbers and give the solution as well.

iii) Prove, that:

$$\forall y \in \mathbb{R}\ \ \exists!\, x \in \mathbb{R} : \mathrm{sh}\, x = y.$$

What is x here.

7. Consider the functions f and g. Find the compositions $f \circ g$ and $g \circ f$:

(a) $f(x) := [x]\ \ (x \in \mathbb{R});\ \ g(x) := \dfrac{1}{x}\ \ (x \in \mathbb{R} \setminus \{0\});$

(b) $f(x) := \mathrm{Sign}\,(x)\ \ (x \in \mathbb{R});\ \ g(x) := \ln(2-x)\ \ (x \in (-\infty; 2))$;

(c) $f(x) := e^x\ \ (x \in \mathbb{R});\ \ g(x) := \ln x\ \ (x \in \mathbb{R}^+)$. Is $f \circ g = g \circ f$?

(d)

$$f(x) = \begin{cases} 0, & \text{ha } x \in (-\infty; 0); \\ 2x, & \text{ha } x \in [0; +\infty); \end{cases} \quad ; \quad g(x) = \begin{cases} 1-x^2, & \text{ha } x \in (-\infty; 1); \\ x-1, & \text{ha } x \in [1; +\infty). \end{cases}$$

8. Consider the given functions f, g and h. Find $\min\{f, g\}$; $\max\{f; g\}$ and $\min/\max\{f; g; h\}$. By definition

$$D_{\min\{f,g\}} = D_f \cap D_g, \quad \min\{f, g\}(x) := \min\{f(x), g(x)\}(x \in D_{\min\{f,g\}} \neq \emptyset).$$

Similarly

$$D_{\max\{f,g\}} = D_f \cap D_g, \quad \max\{f, g\}(x) := \max\{f(x), g(x)\}(x \in D_{\max\{f,g\}} \neq \emptyset).$$

(a) $f(x) := |x|\ \ (x \in \mathbb{R});\ \ g(x) := |1-|x||\ \ (x \in \mathbb{R})$;

(b) $f(x) := \dfrac{1}{x}\ \ (x \in \mathbb{R} \setminus \{0\});\ \ g(x) := 2x\ \ (x \in \mathbb{R})$;

(c) $f(x) := 1\ \ (x \in \mathbb{R});\ \ g(x) := x\ \ (x \in \mathbb{R});\ \ h(x) := x^2\ \ (x \in \mathbb{R})$.

$\boxed{\text{Other types}}$

9. Prove, that the function

$$f : (0; +\infty) \longrightarrow \mathbb{R} \qquad f(x) := x + \frac{1}{x}\ \ (x > 0)$$

is strictly monoton decreasing on the interval $(0; 1]$ and is strictly monotone increasing on the interval $[1; +\infty)$.

10. Prove, that the function

$$f : \mathbb{R} \longrightarrow \mathbb{R} \qquad f(x) := \frac{e^x - 1}{e^x + 1}\ \ (x \in \mathbb{R})$$

is an odd function $(\forall x \in D_f = \mathbb{R} :\ f(-x) = -f(x))$ and it is bounded as well.

11. Find all the functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the following equation:

$$2 \cdot f(x) + 3 \cdot f(1-x) = 4x - 1\ \ (\ \forall\, x \in \mathbb{R}).$$

1.2.2. Homework and more exercises to practice

Domain of definition

1. Find the largest possible set $D \subset \mathbb{R}$ for which the following expressions define a function f:

   (a) $f(x) := \dfrac{\sqrt{\sqrt[3]{x} - 2}}{1 - [x/9]} \quad (x \in D)$;

   (b) $f(x) := \log_{3+x}(x^2 - 1) \quad (x \in D)$;

   (c) $f(x) := \sqrt[3]{\dfrac{x}{1 - |x|}} \quad (x \in D)$;

   (d) $f(x) := \dfrac{\ln(4 - x^2) + \sqrt{1 - x}}{e^x - e^{-x}} \quad (x \in D)$;

   (e) $f(x) := \sqrt{\ln(\cos x)} + \dfrac{1}{\sin x} \quad (x \in D)$.

Equality of functions

2. Is $f = g$, if:

   (a) $f(x) := \sqrt{x^4 + 2x^2 + 1} \ (x \in \mathbb{R})$; $g(x) := 1 - x^3 + (x^2 - 3) \cdot (x + 1) \ (x \in \mathbb{R})$?

   (b) $f(x) := \sqrt{x^4 - 2x^2 + 1} \ (x \in \mathbb{R})$; $g(x) := (x^2 - 1) \cdot \operatorname{sign}(1 - |x|) \ (x \in \mathbb{R})$?

   (c) $f(x) := \dfrac{x^2 - 1}{x + 1} \ (x \in \mathbb{R} \setminus \{-1\})$; $g(x) := x - 1 \ (x \in \mathbb{R})$?

   (d) $f(x) := \dfrac{x^3 - 1}{x - 1} \ (x \in \mathbb{R} \setminus \{1\}) \wedge f(1) := 3$; $g(x) := (x + 1)^2 - x \ (x \in \mathbb{R})$?

   (e) $f(x) := \ln|x| \ (x \in (-\infty; 0) \cup (0; +\infty))$; $g(x) := \ln(-x) \ (x \in (-\infty; 0)$?

   (f) $f(x) := \ln \dfrac{x + 1}{x} \ (x \in (-\infty; -1) \cup (0; +\infty))$;

   $g(x) := \ln(x + 1) - \ln x \ (x \in (0; +\infty))$? Is it true, that$f|_{(0;+\infty)} = g$?

   (g) $f(x) := \ln(\cos^2 x) \ (x \in (-\pi/2; \pi/2))$;

   $g(x) := \ln(1 + \sin x) + \ln(1 - \sin x) \ (x \in (-\pi/2; \pi/2))$?

   What is the largest possible set on which we can extend f and g so, that $f = g$?

   (h) $f(x) := \sin^2 x + \cos^2 x \ (x \in \mathbb{R})$; $g(x) := 1 \ (x \in \mathbb{R})$?

   (i) $f(x) := \cos(2x) \ (x \in \mathbb{R})$;

   $g(x) := (1 - \operatorname{tg}^2 x) \cdot \cos^2 x \ (x \in \mathbb{R} \setminus \{\pi/2 + k\pi \mid k \in \mathbb{Z}\})$?

(j) $f(x) := -\cos(2x)\ \ (x \in (0;\pi));\ \ g(x) := (1-\operatorname{ctg}^2 x)\cdot \sin^2 x\ \ (x \in (0;\pi))$?

(k) $f(x) := \operatorname{tg} x + \operatorname{ctg} x\ \ (x \in \mathbb{R}\setminus \{k\pi/2 \mid k \in \mathbb{Z}\})$;

$$g(x) := \frac{2}{\sin(2x)}\quad (x \in \mathbb{R}\setminus \{k\pi/2 \mid k \in \mathbb{Z}\})?$$

(l) $f(x) := \sqrt{1+\cos x}\ \ (x \in \mathbb{R});\ \ g(x) := \sqrt{2}\cdot \cos \frac{x}{2}\ \ (x \in \mathbb{R})$?

What relation is between f and g?

In case (a),(b),(c),(d),(e),(h),(i),(j) and (l) draw the graphs of f and g as well.

Transformations and graphs

3. Draw the graphs of the following functions and also give the transformational steps as well:

(a) $f(x) := 1-(2x-4)^2\quad (x \in \mathbb{R})$;

(b) $f(x) := |3x-x^2-2|\quad (x \in \mathbb{R})$;

(c) $f(x) := x^2+x+1\quad (x \in \mathbb{R})$;

(d) $f(x) := 9-x^3+6x^2-12x\quad (x \in \mathbb{R})$;

(e) $f(x) := |x^3-1|\quad (x \in \mathbb{R})$;

(f) $f(x) := |x^3|-1\quad (x \in \mathbb{R})$;

(g) $f(x) := ||x-2|-1|\quad (x \in \mathbb{R})$;

(h) $f(x) := x\cdot|x|\quad (x \in \mathbb{R})$;

(i) $f(x) := \dfrac{x+3}{x+1}\quad (-1 \neq x \in \mathbb{R})$;

(j) $f(x) := \dfrac{x-4}{3x-3}\quad (1 \neq x \in \mathbb{R})$;

(k) $f(x) := \dfrac{x}{1-|x|}\quad (x \in \mathbb{R}\setminus\{\pm 1\})$;

(l) $f(x) := \dfrac{\sqrt{16x-4}}{2}+1\quad \left(\dfrac{1}{4} \leq x \in \mathbb{R}\right)$;

(m) $f(x) := 1-\sqrt{x-1}\quad (x \in [1+\infty))$;

(n) $f(x) := \sqrt{|x-1|}\ \ (x \in \mathbb{R})$;

(o) $f(x) := |\sqrt{|x|}-1|\ \ (x \in \mathbb{R})$;

(p) $f(x) := \cos\left(x-\dfrac{\pi}{4}\right)+\cos\left(x+\dfrac{\pi}{4}\right)\quad (x \in \mathbb{R})$;

(q) $f(x) := \sin|x-\pi|+\sin|x+\pi|\quad (x \in \mathbb{R})$;

(r) $f(x) := \sin x\cos x+\dfrac{1}{2}\cdot\cos^2 x-\dfrac{1}{2}\cdot\sin^2 x\quad (x \in \mathbb{R})$;

(s) $f(x) := \sin^4 x + \dfrac{1}{2} \cdot \cos(2x) \quad (x \in \mathbb{R})$;

(t) $f(x) := \operatorname{ctg}\left(\dfrac{\pi}{2} + x\right) \quad (x \in (-\pi/2; \pi/2))$;

(u) $f(x) := e^{|x|} \quad (x \in \mathbb{R})$;

(v) $f(x) := |\ln(1+x)| \quad (x \in (-1; +\infty))$;

(w) $f(x) := \lg \dfrac{100}{x-1} \quad (x \in (1; +\infty))$;

(x) $f(x) := \operatorname{Sign}(x-1) + x \cdot \operatorname{Sign}(x-2) \quad (x \in \mathbb{R})$;

(y)
$$D(x) := \begin{cases} x, & \text{if } x \in \mathbb{R} \cap \mathbb{Q}; \\ x^2, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Operations with functions

4. For the following functions f and g find function h. Where is possible also draw the graphs of $f; g; h$:

(a) $f(x) := x - 1 \ (x \in \mathbb{R}); \ g(x) := \cos x \ (x \in \mathbb{R})$;

Find $h := f + g; \ h := f - g; \ h := f \cdot g; \ h := \dfrac{f}{g}; \ h := \dfrac{g}{f}$?

(b) $f(x) := -D(x) \ (x \in \mathbb{R}); \ g(x) := D(-x) \ (x \in \mathbb{R})$;

Find $h := f + g; \ h := f - g; \ h := f \cdot g; \ h := \dfrac{f}{g}$?

Here D denotes the Dirichlet function.

(c) $f(x) := |x - 1| \ (x \in \mathbb{R}); \ g(x) := 1 - x \ (x \in \mathbb{R})$;

Find $h := f + g; \ h := f - g; \ h := f \cdot g; \ h := \dfrac{f}{g}; \ h := \dfrac{g}{f}$?

(d) $f(x) := \ln x \ (x \in (0; +\infty)); \ g(x) := \ln x^2 \ (0 \neq x \in \mathbb{R})$;

Find $h := f + g; \ h := f - g; \ h := f \cdot g; \ h := \dfrac{f}{g} \ h := \dfrac{g}{f}$?

(e) $f(x) := \sqrt{x-1} \ (x \in [1; +\infty)); \ g(x) := \sqrt{1-x} \ (x \in (-\infty; 1])$;

Find $h := f + g; \ h := f - g; \ h := f \cdot g; \ h := \dfrac{f}{g}; \ h := \dfrac{g}{f}$?

(f) $f(x) := \dfrac{1}{e^x} \ (x \in \mathbb{R}); \ g(x) := e^x \ (x \in \mathbb{R})$;

Find $h := f + g; \ h := f - g; \ h := f \cdot g; \ h := \dfrac{f}{g}; \ h := \dfrac{g}{f}$?

(g)

$$f(x) = \begin{cases} x \cdot (x-1), & \text{ha } x \in \mathbb{Q}; \\ \dfrac{x^2}{2}, & \text{ha } x \in \mathbb{R} \setminus \mathbb{Q}; \end{cases} \quad ; \quad g(x) = \begin{cases} x, & \text{ha } x \in \mathbb{Q}; \\ \dfrac{x^2}{2}, & \text{ha } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Find: $h := f \pm g$; $h := f \cdot g$; $h := \dfrac{f}{g}$; $h := \dfrac{g}{f}$?

5. For the following functions f and g give $f \circ g$ and $g \circ f$ and sketch their graphs too:
   (a) $f(x) := |x| \ (x \in \mathbb{R})$; $g(x) := \sin x \ (x \in \mathbb{R})$;
   (b) $f(x) := \sqrt{x} \ (x \in [0; +\infty))$; $g(x) := x^2 \ (x \in \mathbb{R})$. Is it true, that $f \circ g = g \circ f$?
   (c) $f(x) := [x] \ (x \in \mathbb{R})$; $g(x) := \cos(\pi \cdot x) \ (x \in \mathbb{R})$;
   (d) $f(x) := \dfrac{x}{1+|x|} \ (x \in \mathbb{R})$; $g(x) := \dfrac{x}{1-|x|} \ (x \in (-1; 1))$. Is it true, that $f \circ g = g \circ f$?
   (e)
   $$f(x) = \begin{cases} x^2 + 6x, & \text{if } x \in (-\infty; -3); \\ -2x - 5, & \text{if } x \in [-3; +\infty); \end{cases} \quad ; \quad g(x) = \begin{cases} 5x - 2, & \text{if } x \in (-\infty; 1]; \\ x^2 - 2x + 4, & \text{if } x \in (1; +\infty). \end{cases}$$

6. For the following functions $f; g$ and h find $\min\{f, g\}$; $\max\{f; g\}$ and $\min/\max\{f; g; h\}$. Draw their graphs too:
   (a) $f(x) := |x| \ (x \in \mathbb{R})$; $g(x) := \sqrt{1 - x^2} \ (x \in [-1; 1])$;
   (b) $f(x) := \sqrt{|x|} \ (x \in \mathbb{R})$; $g(x) := \left|\dfrac{1}{x}\right| \ (x \in \mathbb{R} \setminus \{0\})$;
   (c) $f(x) := x \ (x \in \mathbb{R})$; $g(x) := x^2 \ (x \in \mathbb{R})$; $h(x) := x^3 \ (x \in \mathbb{R})$.

$\boxed{\text{Other types}}$

7. Consider the following function:
   $$f(x) = \frac{3+x}{3-x} \cdot \log_{x^2-x-2}(9 - x^2) \quad (x \in D),$$
   where $D \subseteq \mathbb{R}$ denotes the largest real subset, for which f is well-defined. Give the set D and then solve the inequality $f(x) > 0$ on it.

8. Consider the function
   $$f : \mathbb{R} \longrightarrow \mathbb{R} \qquad f(x) := \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad (x \in \mathbb{R}).$$
   Is it an odd or an even function? Is it bounded?

9. Find all the functions $f : \mathbb{R} \setminus \{0, 1\} \longrightarrow \mathbb{R}$ for which we have:

$$f(x) + f\Big(1 - \frac{1}{x}\Big) = 1 + x \quad (\,\forall\, x \in \mathbb{R} \setminus \{0; 1\}).$$

10. Find all the functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ which satisfy the following condition:

$$f(x + y) - f(x - y) = 4xy \quad (\,\forall\, x, y \in \mathbb{R}).$$

11. Consider the parameter $a \in [0; +\infty)$ and the function $f : (a; +\infty) \longrightarrow \mathbb{R}$ so, that $g(x) := \frac{1}{x} \cdot f(x)$ $(x \in (a; +\infty))$ is a monoton decreasing function. Prove, that the function f has the following property (is subadditive):

$$f(x + y) \leq f(x) + f(y) \;\; (\,\forall\, x, y \in (a; +\infty)).$$

12. Consider the function $f(x) := ax + b$ $(x \in \mathbb{R})$, where a and b are real parameters. Give the following function:

$$g := f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n-\text{times}} \;\; (1 \leq n \in \mathbb{N}).$$

2. Image, range of values, invertible functions

2.1. Theory

From now on suppose that $\emptyset \neq A, B$ are arbitrary nonempty sets.

$\boxed{\text{Image, range of values}}$

Definition: Let $f : A \longrightarrow B$ be a function and $C \subseteq A$ a given set. We call the *image* of C by the function f, in notation $f[C]$ the following subset of B:

$$f[C] := \{f(x) \mid x \in C\} \subseteq B.$$

Remarks:

1. We define $f[\emptyset] := \emptyset$.

2. It is clear, that the range of values R_f of a function f is the image of its domain D_f by f, so
$$R_f = f[D_f].$$

For example:

1. If $f(x) := x^2 \ (x \in \mathbb{R})$ and $C := [-1; 2]$, what is the image-set $f[C]$?

   Solution: According to the definition
   $$f[[-1; 2]] = \{f(x) \mid x \in [-1; 2]\} = \{x^2 \mid -1 \leq x \leq 2\} = (\star) = [0; 4],$$
   where $(\star)$ stands for the following deduction:
   $$\text{If } 0 \leq x \leq 2 \implies 0 \leq x^2 \leq 4;$$
   $$\text{If } -1 \leq x \leq 0 \iff 0 \leq -x \leq 1 \implies 0 \leq x^2 \leq 1.$$
   We get that:
   $$\text{If } x \in [-1; 2] \implies x^2 \in [0; 4],$$
   which means that:
   $$f[[-1; 2]] \subseteq [0; 4].$$
   For the reversed order
   $$[0; 4] \subseteq f[[-1; 2]]$$
   we need to prove that:
   $$\forall\ y \in [0; 4] :\ y \in f[[-1; 2]] \iff \forall\ y \in [0; 4]\ \exists\, x \in [-1; 2] :\ x^2 = y.$$

For this reason fix a value $y \in [0;4]$. We want to find an $x \in [-1;2]$, which satisfies the following:

$$x^2 = y \in [0;4] \iff x = \pm\sqrt{y}.$$

It is clear that

$$x = \sqrt{y} \in [0;2] \subseteq [-1;2] \text{ will do.}$$

We have also proved that:

$$[0;4] \subseteq f[[-1;2]].$$

2. Let $f(x) := x^2 \ (x \in \mathbb{R})$. Find the range of values R_f.

Solution: By definition

$$R_f = f[D_f] = \{f(x) \mid x \in D_f\} = \{x^2 \mid x \in \mathbb{R}\} \subseteq [0;+\infty),$$

since

$$\forall\, x \in \mathbb{R} : y = x^2 \geq 0.$$

Let's check now the reversed order:

$$[0;+\infty) \subseteq R_f.$$

We have to chek that:

$$\forall\ y \in [0;+\infty) :\ y \in R_f \iff \forall\ y \in [0;+\infty)\ \exists\, x \in \mathbb{R} :\ x^2 = y.$$

We take now a fixed value $y \in [0;+\infty)$ and want to find an $x \in \mathbb{R}$ for which we have:

$$x^2 = y \in [0;+\infty) \iff x = \pm\sqrt{y} \in \mathbb{R}.$$

As we can see above there are two real numbers x satisfying our wish, so

$$[0;+\infty) \subseteq R_f.$$

These two cases mean that:

$$R_f = [0;+\infty).$$

Pre-image or reversed image

Definition: Let $f : A \longrightarrow B$ be a function and $D \subseteq B$ be a given set. The *pre-image* or the *reversed image* set $f^{-1}[D]$ of D by f is the following subset of A:

$$f^{-1}[D] := \{x \in D_f \mid f(x) \in D\} \subseteq A.$$

Remarks:

1. Consider by definition $f^{-1}[\emptyset] := \emptyset$.

2. It is also clear, that the domain D_f of definition for f is the pre-image of R_f by f, so
$$D_f = f^{-1}[R_f].$$

For example:

1. Consider again the quadratic function
$$f(x) := x^2 \ \ (x \in \mathbb{R}) \ \text{ and the set } D := [1;2].$$
Find the pre/reversed image set $f^{-1}[D]$.

Solution: By definition we have:
$$f^{-1}[[1;2]] = \{x \in D_f \mid f(x) \in [1;2]\} = \{x \in \mathbb{R} \mid 1 \le x^2 \le 2\} = (\star) =$$
$$= [-\sqrt{2};-1] \cup [1;\sqrt{2}],$$
where $(\star)$ for the following solution steps:
$$1 \le x^2 \le 2 \iff (x^2 - 1 \ge 0 \ \wedge \ x^2 - \sqrt{2} \le 0) \iff x \in [-\sqrt{2};-1] \cup [1;\sqrt{2}].$$

2. Let now $g(x) := \dfrac{1}{x}$ $(x \in \mathbb{R} \setminus \{0\})$ be another function. Find the pre-image set $g^{-1}[(-1;1]]$.

Solution: using the definition again:
$$g^{-1}[(-1;1]] = \{x \in D_f \mid g(x) \in (-1;1]\} = \left\{x \in \mathbb{R} \setminus \{0\} \mid -1 < \frac{1}{x} \le 1\right\} = (\star) =$$
$$= (-\infty;-1) \cup [1;+\infty),$$
where $(\star)$ denotes the following solution:
$$-1 < \frac{1}{x} \le 1 \ \ (x \in (-\infty;0) \cup (0;+\infty)) \iff$$
$$\iff ((x < 0 \Longrightarrow x < -1 \ \wedge \ x \le 1) \ \vee \ (x > 0 \Longrightarrow x > -1 \ \wedge \ x \ge 1)) \iff$$
$$\iff x < -1 \ \vee \ x \ge 1.$$

Invertible functions and inverse function

Definition: Let $f : A \longrightarrow B$ be a function. We say, that f is *invertible* or *injective* or it is a *one-to-one* function, if:
$$\forall\, x,t \in D_f : x \ne t \implies f(x) \ne f(t).$$

Remarks:

1. Saying it in words f is invertible, if and only if for all two different values from D_f their images are also different.
2. In practice we use many times the logical "reversed" property, so:
$$f \quad \text{is} \quad \text{injective} \quad \Longleftrightarrow \quad \forall\ x,t \in D_f : f(x) = f(t) \quad \Longrightarrow \quad x = t.$$

For example:

1. Consider the function
$$f(x) := 2x - 7 \ \ (x \in \mathbb{R}).$$
Is f injective?

   Solution: Let $x \neq t \in \mathbb{R}$ be two different arbitrary points in the domain of f. Evaluate their difference:
$$f(x) - f(t) = (2x - 7) - (2t - 7) = 2 \cdot (x - t) \neq 0 \Longrightarrow f(x) \neq f(t),$$
which means that f is injective.
2. Let f be
$$f(x) := \sqrt{9 - x^2} \ \ (x \in [-3; 3]).$$
Is it invertible f now?

   Solution: We can easily observe now, that:
$$-1 \neq 1 \in [-3; 3] \ \ \wedge \ \ f(-1) = f(1) = \sqrt{8} \ \ \Longrightarrow f \quad \text{is not injective.}$$
3. Change now the domain of f from the previous exercise like follows:
$$f(x) := \sqrt{9 - x^2} \ \ (x \in [0; 3]).$$
Is this function f invertible/injective now?

   Solution: Let $x, t \in [0; 3]$ and suppose that $f(x) = f(t)$. Then we have:
$$\sqrt{9 - x^2} = \sqrt{9 - t^2} \quad \Longrightarrow \quad 9 - x^2 = 9 - t^2 \quad \Longleftrightarrow \quad x^2 - t^2 = 0 \quad \Longleftrightarrow$$
$$\Longleftrightarrow \quad (x - t) \cdot (x + t) = 0 \Longrightarrow x = t,$$
because
$$\text{If} \quad x, t \in [0; 3] \quad \Longrightarrow \quad x + t \in [0; 6] \ \ \wedge \ \ (x + t = 0 \quad \Longleftrightarrow \quad x = t = 0).$$
We have proved that f is injective.

Definition: Let $f : A \longrightarrow B$ be an invertible function. In this case the *inverse or reversed function* f^{-1} is defined as:
$$D_{f^{-1}} := R_f \ \ \wedge \ \ \forall\, y \in D_{f^{-1}} : \ \ f^{-1}(y) := x \quad \Longleftrightarrow \quad f(x) = y.$$

Remarks:

1. It can be seen that $R_{f^{-1}} = D_f$.
2. In practice finding the explicite form for $f^{-1}(y)$ means to solve the equation $f(x) = y$ with a given value $y \in R_f$ and the unknown value of x from D_f (so we express x by y).

For example:

1. Let
$$f(x) := 2x - 7 \;\; (x \in \mathbb{R})$$
be our invertible function. Give the reversed function f^{-1}.

Solution: As we have seen before f is really injective. As a first step let's find the range of values R_f of f:
$$R_f = \{f(x) \mid x \in D_f\} = \{2x - 7 \mid x \in \mathbb{R}\} = (\star) = \mathbb{R},$$
where $(\star)$ stays for:
$$\forall x \in \mathbb{R} : f(x) = 2x - 7 \in \mathbb{R}, \;\text{ so }\; R_f \subseteq \mathbb{R}$$
and for the reversed order:

If $\;y \in \mathbb{R} \Longrightarrow \exists\, x \in \mathbb{R} : \;\; y = 2x - 7 \;\; \Longleftrightarrow \;\; \exists\; x = \dfrac{y+7}{2} \in \mathbb{R} : \;\; f(x) = y, \;$ so $\;\mathbb{R} \subseteq R_f$.

We have proved that:
$$D_{f^{-1}} = R_f = \mathbb{R} \;\; \wedge \;\; f^{-1}(y) = \frac{y+7}{2} \quad (y \in \mathbb{R} = D_{f^{-1}})\,.$$

2. Check if the following function is invertible or not, and give its inverse f^{-1} (find $D_{f^{-1}}, R_{f^{-1}}$ and a formula for $f^{-1}(y)$ if $y \in D_{f^{-1}}$):
$$f(x) = \frac{3x+2}{x-1} \quad (x \in (1; +\infty)).$$

• We need to check the statement:
$$\forall\, x, t \in D_f : \; x \neq t \Longrightarrow f(x) \neq f(t),$$
or:
$$\text{if } x, t \in D_f \;\; : f(x) = f(t) \Longrightarrow x = t.$$

Solution 1:

Consider $x, t \in D_f = (1; +\infty)$ to be different points, so $x \neq t$ and evaluate the difference of the outputs:
$$f(x) - f(t) = \frac{3x+2}{x-1} - \frac{3t+2}{t-1} = \frac{(3x+2)\cdot(t-1) - (3t+2)\cdot(x-1)}{(x-1)\cdot(t-1)} =$$

$$= \frac{5 \cdot (t - x)}{(x-1) \cdot (t-1)} \neq 0, \text{ it } x \neq t \Longrightarrow f(x) \neq f(t),$$

which means, that f is invertible (or injective).

Solution 2:

In our second formulation: if $x, t \in D_f = (1; +\infty)$ two inputs, for which: $f(x) = f(t)$, then:

$$\frac{3x+2}{x-1} = \frac{3t+2}{t-1} \Longrightarrow (3x+2) \cdot (t-1) = (3t+2) \cdot (x-1) \Longrightarrow$$

$$\Longrightarrow 3xt - 3x + 2t - 2 = 3tx - 3t + 2x - 2 \Longrightarrow 5x = 5t \Longrightarrow x = t,$$

so f is invertible.

- Find the inverse (or reversed) function f^{-1}, by the following steps:
  - $*$ $D_{f^{-1}} = R_f$
  - $*$ if $x \in D_{f^{-1}}$, then $f^{-1}(x) = y \iff y = f(x)$
  - $*$ $R_{f^{-1}} = D_f = (1; +\infty)$.

Evaluate the range of values of f first. By definition:

$$R_f = \{y \in \mathbb{R} \mid \exists\, x \in D_f : y = f(x)\} = \left\{y \in \mathbb{R} \;\middle|\; \exists\, x \in (1; +\infty) : y = \frac{3x+2}{x-1}\right\}.$$

For a fixed $y \in \mathbb{R}$ we have to solve for $x \in (1; +\infty)$ the equation:

$$y = \frac{3x+2}{x-1} \iff y \cdot (x-1) = 3x + 2 \iff (y-3) \cdot x = y + 2.$$

If $3 \neq y \in \mathbb{R}$, then:

$$\exists\, x = \frac{y+2}{y-3} \; (= f^{-1}(y))$$

is a solution of the given equation. We still have to fulfill the condition $x > 1$, so:

$$x = \frac{y+2}{y-3} > 1 \iff \frac{y+2}{y-3} - 1 > 0 \iff \frac{5}{y-3} > 0 \iff y > 3.$$

Finally we get:

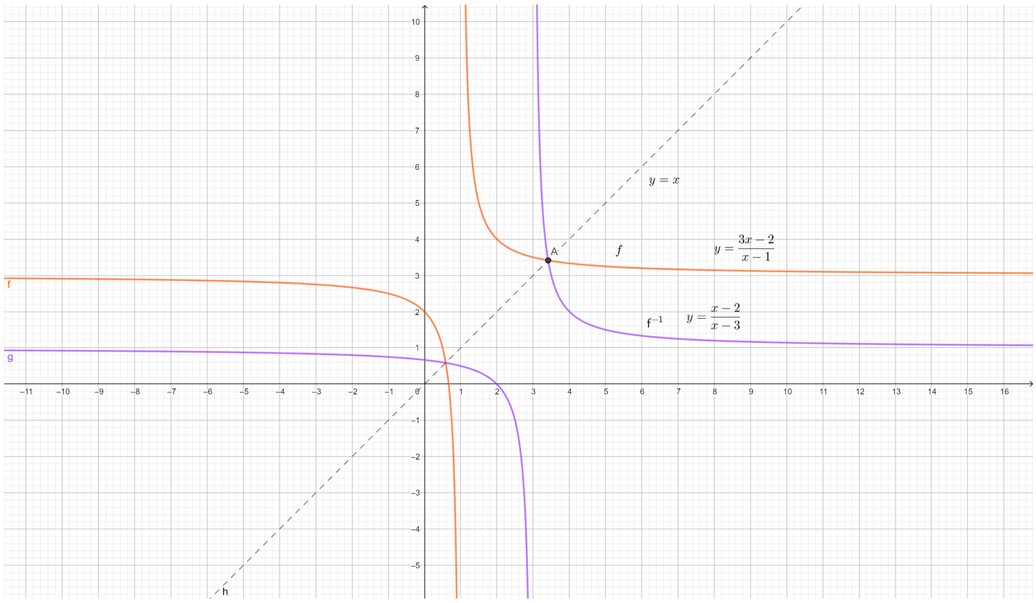
$$D_{f^{-1}} = R_f = (3; +\infty) \;\wedge\; f^{-1}(y) = \frac{y+2}{y-3} \; (y > 3).$$

Remark: We may first take the following form of f (which is also convenient for sketching the graph of f):

$$y = 3 + 5 \cdot \frac{1}{x-1} \ \ (x > 1) \iff x = 1 + 5 \cdot \frac{1}{y-3} \ \ (y > 3).$$

We also can see from our calculations, that if $D_f = \mathbb{R} \setminus \{1\}$, then$f$ is still invertible, and in this case $D_{f^{-1}} = \mathbb{R} \setminus \{3\}$. The next graph is illustrating this situation.

Here are the graphs of f and f^{-1} in the same coordinate system (on the widest possible range):



3. Examine the following function from point of view of invertibility. When it is injective give the inverse function as well.

$$f(x) = x^2 - 2x + 2 \ \ (x \in (-\infty; 1]).$$

• Invertibility: we need to prove for example, that:

$$\forall\, x, t \in D_f : \ x \neq t \Longrightarrow f(x) \neq f(t).$$

Let $x, t \in D_f = (-\infty; 1]$ and $x \neq t$ two inputs. In this case:

$$f(x) - f(t) = (x^2 - 2x + 2) - (t^2 - 2t + 2) = x^2 - 2x - t^2 + 2t =$$

$$= (x^2 - t^2) - 2 \cdot (x - t) = (x - t) \cdot (x + t - 2) \neq 0,$$

because with the given conditions on x, t we have $x \leq 1$ and $t < 1$ (or conversely), so:

$$x + t - 2 < 1 + 1 - 2 = 0.$$

This implies $f(x) \neq f(t)$, so f is invertible.

- To give f^{-1} we need to find:
  - $D_{f^{-1}} = R_f$
  - if $x \in D_{f^{-1}}$, then $f^{-1}(x) = y \iff y = f(x)$
  - $R_{f^{-1}} = D_f = (-\infty; 1]$.

First find the range of values of f:

$$R_f = \{y \in \mathbb{R} \mid \exists x \in D_f : y = f(x)\} = \left\{y \in \mathbb{R} \mid \exists\, x \in (-\infty; 1] : y = x^2 - 2x + 2\right\}.$$

For a given $y \in \mathbb{R}$, we have to solve the following equation with $x \in (-\infty; 1]$:

$$y = x^2 - 2x + 2 \iff x^2 - 2x + (2 - y) = 0 \iff x_{1,2} = \frac{2 \pm \sqrt{4 - 4 \cdot (2 - y)}}{2} =$$

$$= 1 \pm \sqrt{y - 1} \in \mathbb{R} \iff y - 1 \geq 0 \iff y \geq 1.$$

If $y \in [1; +\infty)$, then the upper equation can be solved on the set of the real numbers, but we still have to satisfy the following condition on x:

$$x_{1,2} = 1 \pm \sqrt{y - 1} \in (-\infty; 1].$$

Assume, that $y \geq 1$ and

$$x_1 = 1 + \sqrt{y - 1} \leq 1 \iff \sqrt{y - 1} \leq 0 \iff y \leq 1,$$

which gives the single $y = 1$ value. For the second root as well:

$$x_2 = 1 - \sqrt{y - 1} \leq 1 \iff \sqrt{y - 1} \geq 0 \iff y \geq 1,$$

which satisfies the condition.

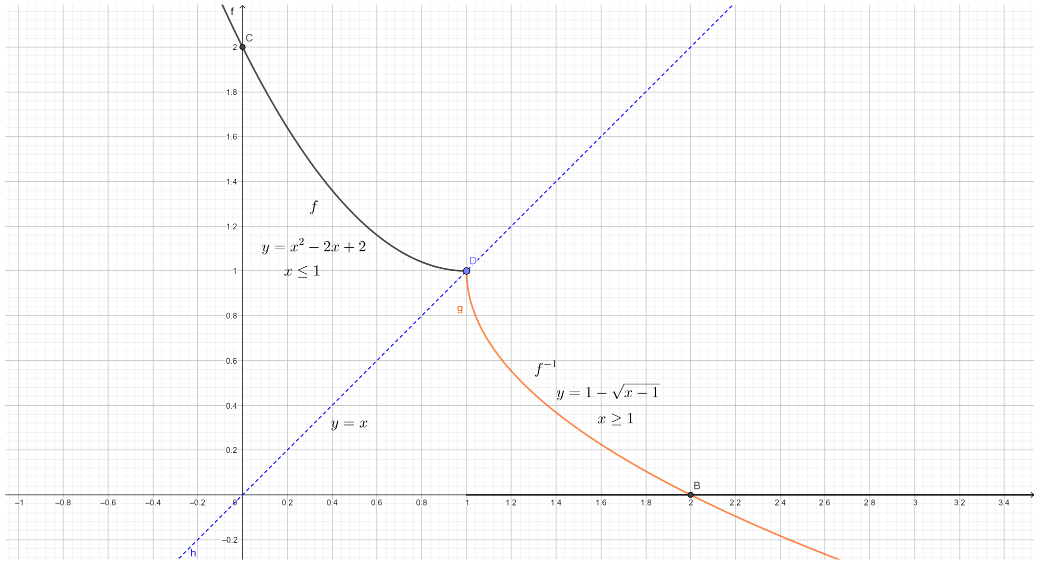
So, when $y = 1$, then $x_1 = x_2 = 1$, and:

$$D_{f^{-1}} = R_f = [1; +\infty) \ \wedge \ f^{-1}(y) = 1 - \sqrt{y - 1} \ \ (y \geq 1).$$

Remark: Sketch the graphs of f, f^{-1} in the same Cartesian coordinate system. The upper evaluations and the graphs can be easily done, by using the perfect square form of f and f^{-1}:

$$y = (x - 1)^2 + 1 \ \ (x \leq 1) \iff x = 1 - \sqrt{y - 1} \ \ (y \geq 1).$$

Here are the graphs of f and f^{-1} in the same coordinate system:



4. Is the following function invertible, and if it is, give f^{-1} as well:

$$f(x) := \frac{1-\sqrt{x}}{1+\sqrt{x}} \quad (x \in [0;+\infty)).$$

Solution: Let $x, t \in [0;+\infty)$ and suppose that $f(x) = f(t)$. In this case:

$$\frac{1-\sqrt{x}}{1+\sqrt{x}} = \frac{1-\sqrt{t}}{1+\sqrt{t}} \iff (1-\sqrt{x})\cdot(1+\sqrt{t}) = (1-\sqrt{t})\cdot(1+\sqrt{x}) \iff \sqrt{x} = \sqrt{t} \implies$$

$$\implies x = t,$$

which shows us that f is injective. What is R_f? Consider the following form of the values $f(x)$:

$$\text{If} \quad x \in [0;+\infty), \quad \text{then}: \quad f(x) = \frac{1-\sqrt{x}}{1+\sqrt{x}} = \frac{2-1-\sqrt{x}}{1+\sqrt{x}} = \frac{2}{1+\sqrt{x}} - 1 > -1 \implies$$

$$\implies R_f \subseteq (-1;+\infty).$$

For the reversed direction let $y \in (-1;+\infty)$ an arbitrary value and we are looking for a value $x \geq 0$, for which we have:

$$f(x) = y \iff \frac{2}{1+\sqrt{x}} - 1 = y \iff \frac{2}{1+\sqrt{x}} = y+1 \iff (y \neq -1) \iff$$

$$\iff \sqrt{x} = \frac{2}{y+1} - 1.$$

This last equation can be solved considering the previous condition $y > -1$ as well, if:

$$\frac{2}{y+1} - 1 \geq 0 \;\Big|\; \cdot (y+1) > 0 \iff 2 \geq y+1 \iff y \leq 1.$$

We've got that:

$$\forall\ y \in (-1;1]\ \exists\ x = \left(\frac{2}{y+1} - 1\right)^2 \in [0;+\infty):\ \ f(x) = y \implies$$

$$\implies (-1;1] \subseteq R_f.$$

Turning back to our previous condition $R_f \subseteq (-1;+\infty)$ we prove that $R_f \subset (-1;1]$ is true as well. For this is enough for example to see that:

$$\forall\ x \in [0;+\infty)\ \frac{2}{1+\sqrt{x}} - 1 \le 1 \iff \forall\ x \in [0;+\infty):\ \ 0 \le \sqrt{x}.$$

So finally:

$$R_f = (-1;1].$$

The reversed function is:

$$D_{f^{-1}} = R_f = (-1;1]\ \wedge\ f^{-1}(y) = \left(\frac{2}{y+1} - 1\right)^2\quad (y \in (-1;1]).$$

2.1.1. Checking questions to the theory and its use

1. Define the *image* set $f[C]$ of a set C by a function f.
2. Define the *pre-image* set $f^{-1}[D]$ of a set D by a function f.
3. When do we call a function injective?
4. Give the definition of the reversed function f^{-1} for an injective function f.
5. Consider the function
$$f(x) := 3x - 1\quad (x \in \mathbb{R}).$$
Find the image set $f[[-3;4)]$.
6. Consider the function
$$f(x) := |x|\quad (x \in \mathbb{R}).$$
What is the image set $f[(-2;1]]$.
7. Consider the function
$$f(x) := 3x - 1\quad (x \in \mathbb{R}).$$
Find $f^{-1}[[-1;5)]$.
8. Consider the function
$$f(x) := |x|\quad (x \in \mathbb{R}).$$
Find $f^{-1}[(1;7]]$.

9. Consider the function
$$f(x) := \ln x \quad (x \in (0; +\infty)).$$
Give $f^{-1}[(-1; 1)]$.

10. Consider the function
$$f(x) := x^2 + x - 3 \quad (x \in \mathbb{R}).$$
Find the sets $f[C]$ and $f^{-1}[C]$, if $C := \{-1\}$.

11. Consider the function
$$f(x) := 1 - 4x \quad (x \in \mathbb{R}).$$
Is f invertible? Give f^{-1}, if it is.

12. Consider the function
$$f(x) := 1 - 2x - x^2 \quad (x \in (-\infty; -1)).$$
Is f injective? Give f^{-1} , if it is.

13. Consider the function
$$f(x) := |x - 1| + (x^2 - 4x + 4) \quad (x \in \mathbb{R}).$$
Is f injective?

14. Consider the function
$$f(x) := 2^x - 1 \quad (x \in \mathbb{R}).$$
Is f invertible? Give f^{-1} if it is.

15. Consider the function
$$f(x) := \ln{(x - 1)} \quad (x \in (1; +\infty)).$$
Is f invertible? Give f^{-1} if it is.

16. Consider the function
$$f(x) := \cos x \quad (x \in \mathbb{R}).$$
Is f invertible? Give f^{-1} if it is.

17. Consider the function
$$f(x) := \frac{1}{1 + e^{-x}} \quad (x \in \mathbb{R}).$$
Is f invertible? Give f^{-1} if it is.

18. Consider the function
$$f(x) := x^3 - 3x^2 + 3x + 2 \quad (x \in \mathbb{R}).$$
Is f invertible? Give f^{-1} if it is.

2.2. Exercises

2.2.1. Exercises for class work

Image, reversed image, range of values

1. Consider the following functions f and sets B, C. Find sets $f^{-1}[B]$ and $f[C]$:
   (a) $f(x) := 2x + 1 \quad (x \in \mathbb{R});\ \ B := [1; 2);\ \ C := (1; 2];$
   (b) $f(x) := 2 - \sqrt{x} \quad (x \in [0; +\infty));\ \ B := [-1; 1];\ \ C := [2; 9];$
   (c) $f(x) := |1 - |x - 2|| \quad (x \in [-1; 4]);\ \ B := [1/4; 1/2);\ \ C := [-1; 2];$
   (d) $f(x) := (\sqrt{2})^{2x+1} \quad (x \in \mathbb{R});\ \ B := [\sqrt{2}; 2);\ \ C := [-1; 1);$
   (e) $f(x) := \dfrac{2 - x}{1 - x} \quad (1 \neq x \in \mathbb{R});\ \ B := [1/2; +\infty);\ \ C := [0; 1) \cup (1; +\infty);$
   (f) $f(x) := [\sin x] \quad (x \in \mathbb{R});\ \ B := [-1; 0];\ \ C := [-1; 0].$

2. Give the range of values R_f for the following functions f:
   (a) $f(x) := 3x + 1 \quad (x \in [-2; 1])$;
   (b) $f(x) := 1 - 2x \quad (-1 \leq x < 3)$;
   (c) $f(x) := |x - 2| \quad (x \in [-1; 4])$.

3. Give the range of values R_f for the following functions f:
   (a) $f(x) := x^2 - 6x + 5 \quad (x \in \mathbb{R});$
   (b) $f(x) := x^2 - 6x + 5 \quad (-1 \leq x \leq 6);$
   (c) $f(x) := 1 - x^2 \quad (-2 \leq x \leq 3).$

4. Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}\ \ f(x) = \dfrac{x^2 - 4x + 3}{x^2 - 2x + 3}$ $(x \in \mathbb{R})$. Give the range of values R_f for f.

5. Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}\ \ f(x) = \dfrac{x^2 + ax + 1}{x^2 - x + 1}$ $(x \in \mathbb{R})$, where $a \in \mathbb{R}$ is a real parameter. For what values of a is the condition $R_f \subset [-3; 2]$ satisfied?

6. Consider the following function f, where $m \in \mathbb{R}$ is a real parameter:

$$f(x) := \begin{cases} 1 - x & \text{, if } x \in (-\infty; -1); \\ 1 - x^2 & \text{, if } x \in [-1; 1]; \\ 1 + x & \text{, if } x \in (1; +\infty). \end{cases}$$

Find $f^{-1}[(m; +\infty)]$ depending on m.

$\boxed{\text{invertible functions, the inverse function}}$

7. Consider the function f:

$$f(x) := \begin{cases} \dfrac{x+1}{x-1}, & \text{if } x \in (1; +\infty); \\ 1, & \text{if } x \in (-\infty; 1]. \end{cases}$$

Give the function $g(x) := f(x+1) - f(x-1) \quad (x \in \mathbb{R})$ and the sets $g[[-1; 3]]$ and $g^{-1}[[-1; 2]]$. Is g invertible? Prove that the function $g|_{(0;+\infty)}$ is invertible and give its inverse!

8. Are the following functions invertible? Give f^{-1} if it exists (by giving $D_{f^{-1}}; R_{f^{-1}}$ and $f^{-1}(x)$ for all $x \in D_{f^{-1}}$):

   (a) $f(x) := 2x - 1 \quad (x \in \mathbb{R})$;

   (b) $f(x) := x^2 - 2x + 2 \quad (x \in (-\infty; 1])$;

   (c) $f(x) := 1 - \sqrt{2 - x} \quad x \in (-\infty; 2]$;

   (d) $f(x) := \dfrac{x+1}{x-1} \quad (x \in (1; +\infty))$;

   (e) $f(x) := \dfrac{1}{1 + x^3} \quad (x \in \mathbb{R} \setminus \{-1\})$;

   (f) $f(x) := |x - 1| + (x + 2)^2 \quad (x \in \mathbb{R})$;

   (g) $f(x) := \dfrac{2x}{1 + x^2} \quad (x \in \mathbb{R})$;

   (h) $f(x) := \dfrac{e^x - e^{-x}}{2} \quad (x \in \mathbb{R})$;

   (i) $f(x) := x \cdot |x| + 2x \quad (x \in \mathbb{R})$.

2.2.2. Homework and more exercises to practice

$\boxed{\text{Image, reversed image, range of values}}$

1. For the following functions f and sets B, C give $f^{-1}[B]$ and $f[C]$:

   (a) $f(x) := 4 - 3x \quad (x \in \mathbb{R});\ B := [-1; 2);\ C := (-1; 2];$

   (b) $f(x) := 1 + \sqrt{1-x} \quad (x \in (-\infty; 1]);\ B := [1/2; 3];\ C := [0; 1/2];$

   (c) $f(x) := |1 - x^2| \quad (x \in \mathbb{R});\ B := [-1; 1/2];\ C := [-2; 3];$

   (d) $f(x) := 3^{1/2-2x} \quad (x \in \mathbb{R});\ B := (1/27; 3];\ C := [-1/4; 1/4);$

   (e) $f(x) := \dfrac{x}{1+x} \quad (-1 \neq x \in \mathbb{R});\ B := [0; +\infty);\ C := (-\infty; -1) \cup (-1; 1];$

   (f) $f(x) := [\cos x] \quad (x \in \mathbb{R});\ B := [0; 1];\ C := [0; \pi].$

2. Give R_f for the following functions:

   (a) $f(x) := -x^2 - x - 1 \quad (x \in \mathbb{R});$

   (b) $f(x) := (x-1) \cdot (3-x) \quad (1 \leq x \leq 4);$

   (c) $f(x) := x^2 - 10x + 27 \quad (x \in [0; 6]).$

3. Give the range of values for the following functions:

   (a) $f(x) := 2x - \sqrt{2} \quad (x \in [-\sqrt{2}; \sqrt{2}]);$

   (b) $f(x) := 1 + 3 \cdot \sqrt{|x-2|} \quad (x \in \mathbb{R}).$

   (c) $f(x) := \sqrt{4x^2 - 1} \quad \left(\dfrac{1}{2} \leq x < 1\right);$

   (d) $f(x) := \lg(2x+1) \quad \left(-\dfrac{1}{4} \leq x < \dfrac{9}{2}\right);$

   (e) $f(x) := \dfrac{2x}{1+x^2} \quad (x \in \mathbb{R});$

   (f) $f(x) := 4^x - 2^x + 1 \quad (x \in \mathbb{R});$

   (g) $f(x) := (\sin x + \cos x)^2 \quad (x \in \mathbb{R});$

   (h) $f(x) := [\sin x] + [\cos x] \quad (x \in [0; 2\pi]).$

4. For what parameters $k \in \mathbb{R}$ will be the range of values R_f equal to the interval $[0; 5]$, if:
$$f(x) := \sqrt{x^2 + 4x + k} \qquad (|x| \leq 3)?$$

5. Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}\ \ f(x) = \dfrac{x^2 - 2x - 3}{x^2 + x + 1}$ $(x \in \mathbb{R})$. Find the set R_f.

6. Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}\ \ f(x) = \dfrac{3x^2 + ax - 1}{x^2 + 1}$ $(x \in \mathbb{R})$, where $a \in \mathbb{R}$ is a real parameter. Find those values of a, for which we have: $R_f = [-3; 5]$.

7. Consider the function:

$$f(x) := \begin{cases} x^2, & \text{if } x \in (-\infty; 0); \\ \sin x, & \text{if } x \in [0; 2\pi]; \\ 2\pi - x, & \text{if } x \in (2\pi; +\infty). \end{cases}$$

Find the following sets:

$$f^{-1}\Big[[-1;1]\Big];\ f\Big[[-1;\pi]\Big];\ f^{-1}\Big[[0;+\infty)\Big];\ f^{-1}\Big[(-\infty;-1]\Big];$$

$$f\Big[[\pi;3\pi]\Big];\ f^{-1}\Big[\{-1;1\}\Big];\ f^{-1}\Big[\{-1/2\}\Big].$$

8. Find the values of the real parameter m so that

$$f^{-1}\Big[(-\infty;0)\Big] = \mathbb{R}$$

for the function:

$$f(x) := (m+1)x^2 + 2mx + 1 \ \ (x \in \mathbb{R}).$$

$\boxed{\text{Invertible functions, the inverse function}}$

9. Check whether the following functions are invertible or not, and in case when they are find f^{-1} (by giving $D_{f^{-1}}; R_{f^{-1}}$ and $f^{-1}(x)$ for all $x \in D_{f^{-1}}$):

   (a) $f(x) := 2 - 5x \quad (x \in \mathbb{R});$

   (b) $f(x) := 1 - 2x - x^2 \quad (x \in [-1; +\infty));$

   (c) $f(x) := x^3 - x \quad (x \in \mathbb{R});$

   (d) $f(x) := \dfrac{x}{1+|x|} \quad (x \in \mathbb{R});$

   (e) $f(x) := \sqrt{x-2} - 1 \quad (x \in [2; +\infty));$

   (f) $f(x) := \sqrt{x^2+2x+5} \quad (x \in \mathbb{R});$

   (g) $f(x) := \sqrt{x^2+2x+5} \quad (x \in (-1; +\infty));$

   (h) $f(x) := \sqrt{x+1} - \sqrt{x} \quad (x \in [0; +\infty));$

   (i) $f(x) := \dfrac{1}{1+\sqrt[3]{x}} \quad (x \in \mathbb{R} \setminus \{-1\});$

   (j) $f(x) := x \cdot \sin x \quad (x \in (-\pi; \pi/2]);$

   (k) $f(x) := \dfrac{x^2+3x}{x^2-2x} \quad (x \in \mathbb{R} \setminus \{0; 2\});$

   (l) $f(x) := \dfrac{e^x + e^{-x}}{2} \quad (x \in (0; +\infty));$

(m) $f(x) := x \cdot |x| - 2x - 8 \quad (x \in \mathbb{R})$.

10. Consider the function $f(x) := ax + b \ (x \in \mathbb{R})$ where a and b real parameters. Find the values of a, b so that f is invertible and $f = f^{-1}$.

11. Prove that if a, b are real parameters so that $ab \neq -4$, then the following function f is invertible and $f = f^{-1}$:

$$f(x) := \frac{2x + a}{bx - 2} \quad (2/b \neq x \in \mathbb{R}).$$

3. Bounded functions, extremal values, limits at $+\infty$

3.1. Theory

Bounded functions

Definition : We say that a function $f \in \mathbb{R} \longrightarrow \mathbb{R}$:

1. is *bounded from below*, if the range of values R_f is bounded from below, which means that:
$$\exists\, k \in \mathbb{R}: \quad f(x) \geq k \quad (\forall\, x \in D_f);$$
2. is *bounded from above*, if the range of values R_f is bounded from above, which means that:
$$\exists\, K \in \mathbb{R}: \quad f(x) \leq K \quad (\forall\, x \in D_f);$$
3. is *bounded*, if the range of values R_f is bounded (from below and from above as well), so
$$\exists\, k, K \in \mathbb{R}: \quad k \leq f(x) \leq K \quad (\forall\, x \in D_f).$$

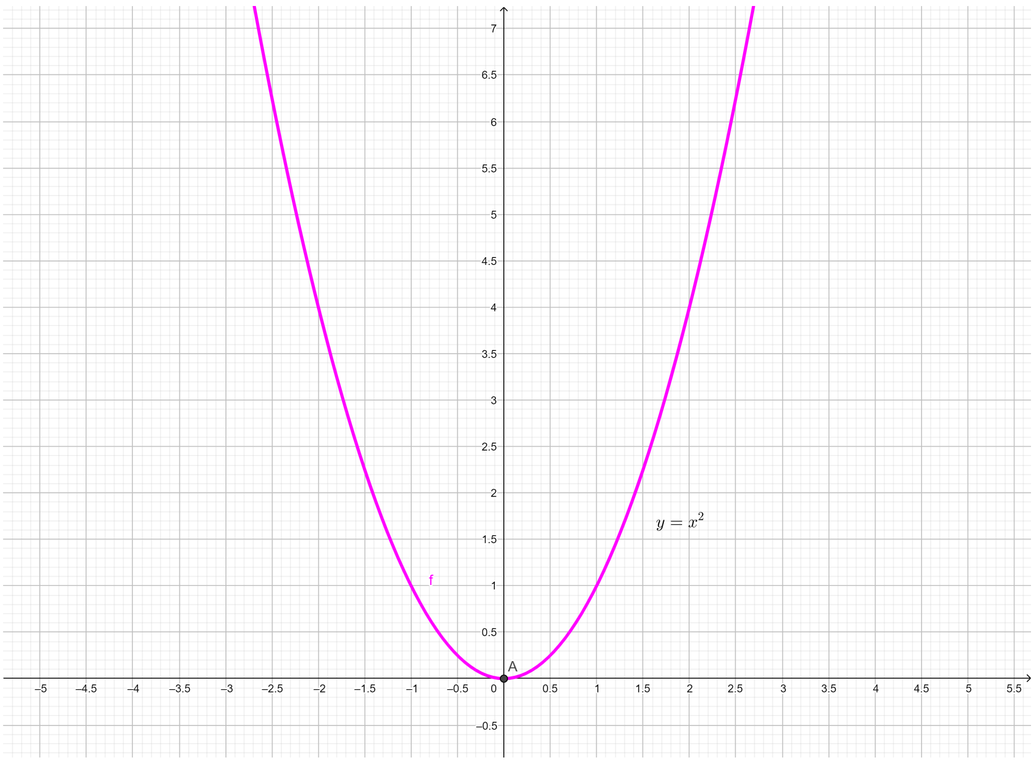
Remarks:

1. The constants k and K are called the lower and upper bounds of f.
2. A given real-real type function f is *not bounded*, if it is not bounded from above or from below or both.

Examples:

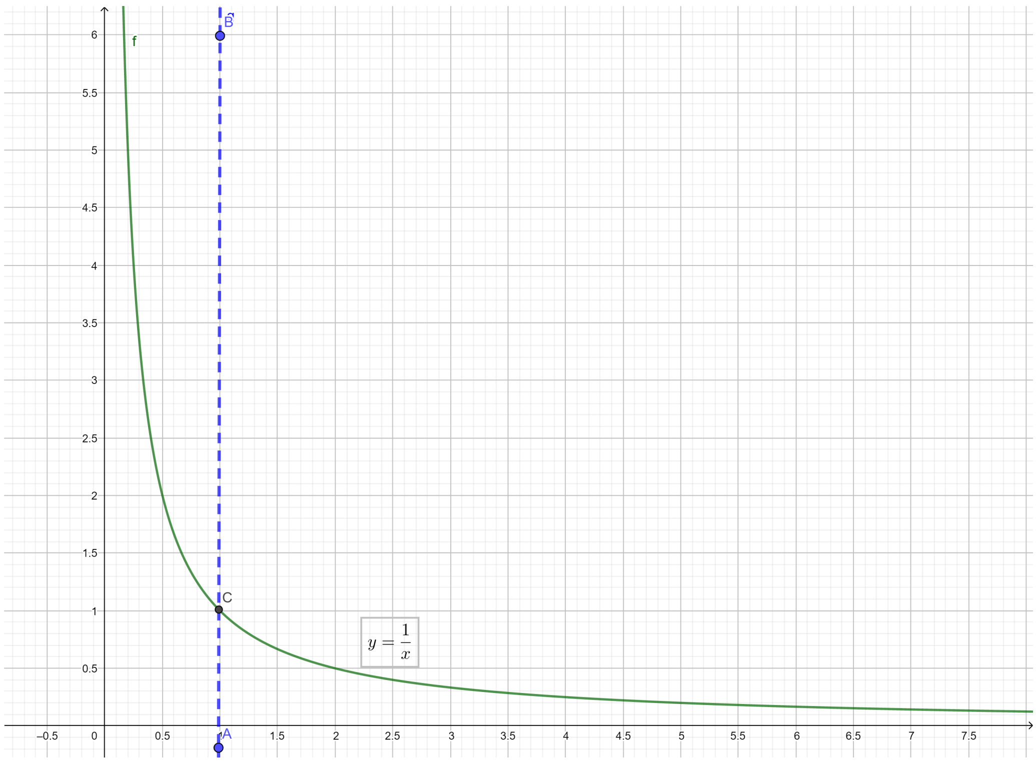
1. Consider the function
$$f(x) := x^2 \;\; (x \in \mathbb{R}).$$
It is easy to see that:
$$\forall\, x \in \mathbb{R}: \quad 0 \leq f(x) = x^2,$$
so f is bounded below and 0 is a lower bound (it is its minimal value as well). We prove that f is not bounded from above. Indeed if we suppose, that f has an upper bound, then:
$$\exists\, K \in [0; +\infty): \quad 0 \leq x^2 \leq K \quad (\forall\, x \in \mathbb{R}),$$
what is false for: $x = \sqrt{K+1} \in \mathbb{R}$.

Here is the graph of our normal-parabola:



2. If
$$f(x) := \frac{1}{x} \quad (x \in (0;1))$$
we can prove that:
$$\forall\, x \in (0;1): \quad 1 < \frac{1}{x},$$
but there is no such a positive number $K > 0$ for which we have:
$$\frac{1}{x} \leq K \quad (x \in (0;1)),$$
because for such a number K the following number $x = \dfrac{1}{K+1} \in (0;1)$ fails to satisfy the upper inequality. We can also observe, that here 1 is a lower bound, but it's not a minimal value of f. So this function is bounded from below, but is not bounded from above.

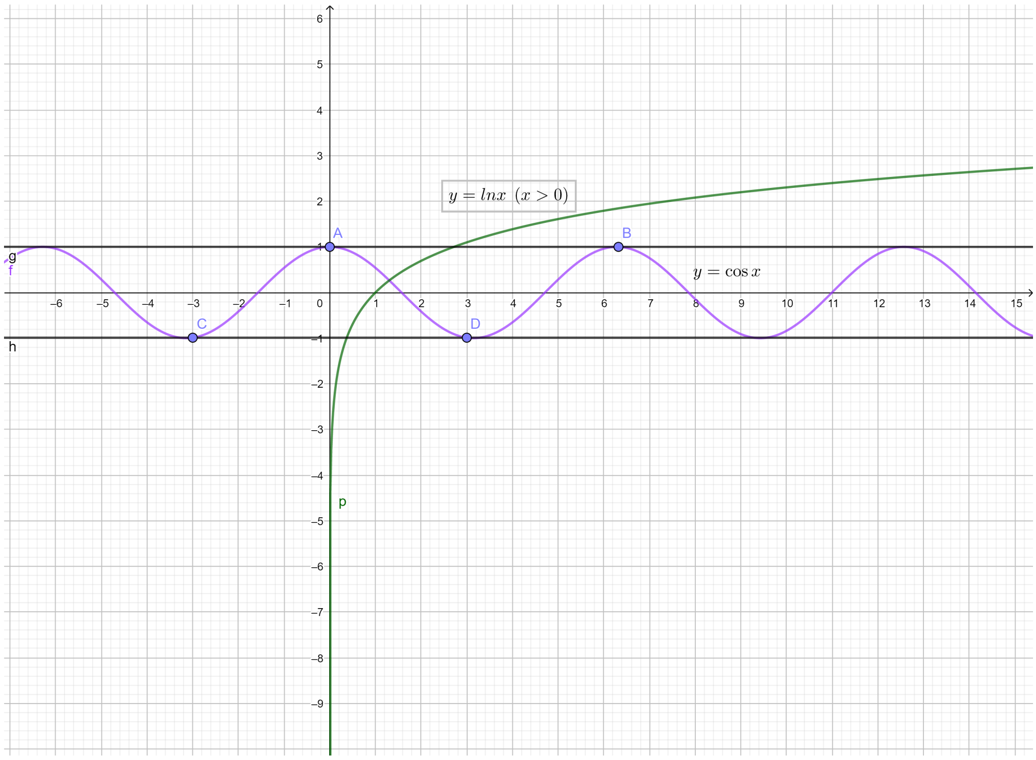
Here is the graph of the normal-hyperbola for $x > 0$:



3. Consider the function
$$f(x) := \cos x \quad (x \in \mathbb{R}).$$
In this case:
$$\forall\, x \in \mathbb{R}: \quad -1 \leq \cos x \leq 1,$$
so f is bounded. We know that -1 and $+1$ are the minimal and maximal values of f. They are at the same time lower and uper bounds as well.

4. If
$$f(x) := \ln x \quad (x \in (0;1)),$$
then we have:
$$\forall\, x \in (0;1): \quad -\infty < \ln x < 0,$$
so f is bounded from above. In this case we don't have a lower bound $k < 0$, because if we would have one, then:
$$\forall\, x \in (0;1): \quad k < \ln x < 0,$$
and the number $x := e^{k-1} \in (0;1)$ would give a contradiction to us.

Here are the graphs of the cosine-function (the purple one) and the ln (natural logarithm-)function (the green one) on their widest possible domain:



Extremal values of a function

Definition : We say that a real-real function $f \in \mathbb{R} \longrightarrow \mathbb{R}$:

1. has a *minimal value*, if the set R_f has a smallest/minimal value, so:

$$\exists\, a \in D_f : \quad f(x) \geq f(a) \quad (\forall\, x \in D_f);$$

2. has a *maximal value*, if the set R_f has a maximal value, so:

$$\exists\, b \in D_f : \quad f(x) \leq f(b) \quad (\forall\, x \in D_f).$$

We say that a and b are the *minimum place* and *maximum place* of f and $f(a), f(b)$ are the corresponding minimal and maximal *values*.

Remark:

1. If a real$-$real function f has extremal value/values (minimal value, maximal value, or both), then these values can be taken in many places. For example the well-known

sin and *cos* functions take their maximal value of 1 and their minimal value of -1 at infinitely many points. Similarly the constant function

$$f(x) := 3 \quad (x \in \mathbb{R})$$

takes its minimal and maximal value of 3 at all real numbers.

Examples: Consider the following functions f and find their minimal and maximal values (if they exist). Also find the places, where these values are taken.

1. Let $f(x) := x^2 \quad (x \in [-1;3])$. It is obvious, that:

   $$\forall\, x \in [0;3] : f(0) = 0 \leq x^2 \leq 9 = f(3),$$

   and

   $$\forall\, x \in [-1;0] : -1 \leq x \leq 0 \iff 0 \leq -x \leq 1 \Longrightarrow 0 \leq x^2 \leq 1 \Longrightarrow$$
   $$\Longrightarrow f(x) \in [0;1].$$

   We can conclude:

   $$\forall\, x \in [-1;3] : f(0) = 0 \leq x^2 \leq 9 = f(3),$$

   so the minimal value is $y = 0$ at $x = 0$ and the maximal value is $y = 9$ at $x = 3$.

2. Let $f(x) := \sqrt{x+1} - \sqrt{x} \quad (x \in [0;1])$. First we make the following steps:

   $$f(x) = \sqrt{x+1} - \sqrt{x} = \frac{1}{\sqrt{x+1} + \sqrt{x}} \quad (x \in [0;1]).$$

   We can conclude now, that f takes the greatest value when x is the smallest possible and conversely:

   $$\forall\, x \in [0;1] : f(1) = \frac{1}{\sqrt{2}+1} \leq \frac{1}{\sqrt{x+1} + \sqrt{x}} = f(x) \leq \frac{1}{\sqrt{0+1} + \sqrt{0}} = 1 = f(0),$$

   so the minimal value is $y = \sqrt{2} - 1$ at $x = 1$ and the maximal value is $y = 1$ at $x = 0$. See the graph of this function (the orange one) on the next image (it is represented on its maximal domain, which is $[0;+\infty)$.

3. Let $f(x) := \dfrac{x}{1+x^2} \quad (x \in \mathbb{R})$. Observe that:

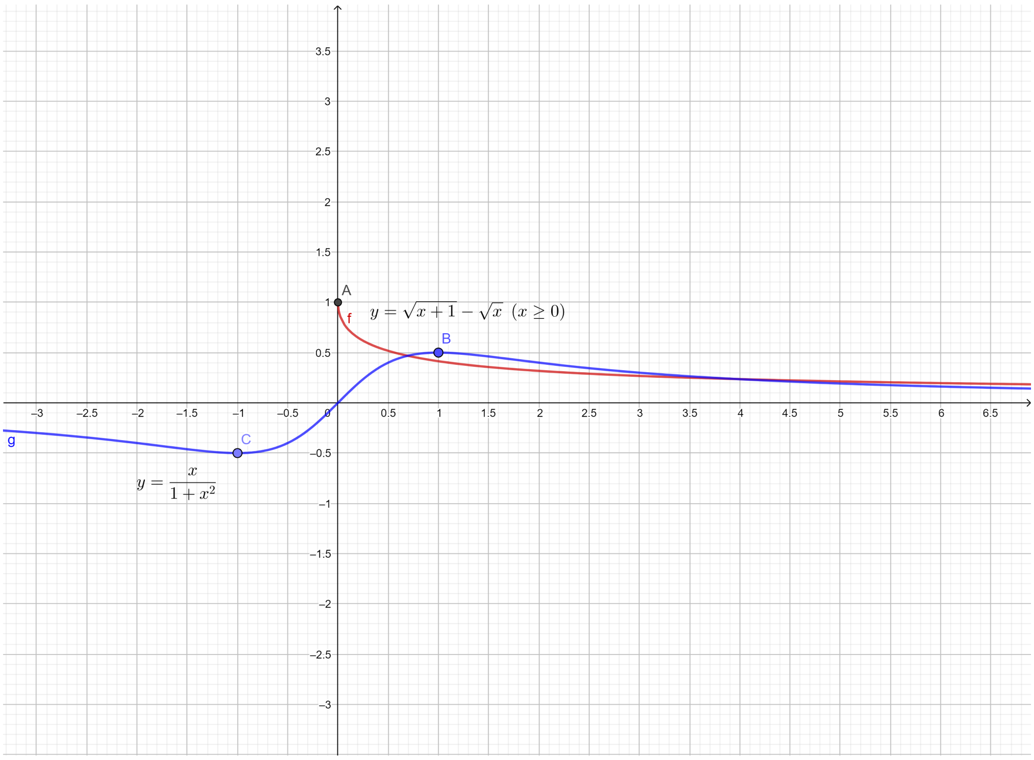
   $$\forall\, x \in \mathbb{R} : \quad f(-1) = -\frac{1}{2} \leq \frac{x}{1+x^2} \leq \frac{1}{2} = f(1),$$

   since:

   $$-\frac{1}{2} \leq \frac{x}{1+x^2} \leq \frac{1}{2} \iff -1 - x^2 \leq 2x \leq 1 + x^2 \iff$$
   $$\iff 0 \leq (x+1)^2 \ \wedge \ 0 \leq (x-1)^2 \quad (\forall\, x \in \mathbb{R}).$$

   So the maximal value of f is now $y = \dfrac{1}{2}$ at $x = 1$ and the minimal value is $y = -\dfrac{1}{2}$ at $x = -1$. We can also see that f is an *odd* function, so its graph is symmetrical on point $(0;0)$. In such cases it is enough to investigate our questions for nonnegative x values of the domain. See the graph of this function (the blue one) on the next image.

Here are the graphs of the given functions in point 2. and 3. respectively (on their maximal domain of definiton):



Order preserving estimates, limits at $+\infty$

In this chapter we want to investigate the behaviour of some real-to-real functions for great enough x values of the domain D_f. We want to define the concept of a limit of a function at $+\infty$. We only consider in this book the three possible cases, namely: finite number/-∞/$+\infty$ as a limit of f at $+\infty$.

For the rest of this chapter we assume that the given functions $f \in \mathbb{R} \to \mathbb{R}$ have the following property:

$$\forall\, P > 0 : \quad (P; +\infty) \cap D_f \neq \emptyset.$$

In other words we say that $+\infty$ is an accumulation-point of the set D_f (in notation $+\infty \in D'_f$). A complete definition of this notion will come in your Analysis courses later.

Def: *(Infinite limit at $+\infty$)* Let $f \in \mathbb{R} \longrightarrow \mathbb{R}$ be a real-to-real function and $+\infty \in D'_f$. We say that f has the limit $+\infty$ at $+\infty$, in notation

$$\lim_{x \to +\infty} f(x) = +\infty,$$

if the following property is fulfilled:

$$\forall\, P > 0\ \exists\, K > 0\, \forall\, x \in (K; +\infty) \cap D_f\ :\ f(x) > P.$$

Examples:

1. Prove by definition that: $\lim\limits_{x\to+\infty} x^2 = +\infty$.

   Solution:

   - What is the definition? The following statement:

   $$\forall\, P > 0 \;\exists\, K > 0 \;: \forall\, x \in (K; +\infty) \cap D_f \;:\; x^2 > P.$$

   Fix a number $P > 0$ and assume that $x > 0$ is also valid. Check when is:

   $$f(x) = x^2 > P > 0.$$

   In this case it is easy to solve it. Since P is a positive number, we can take its square root, so we have:

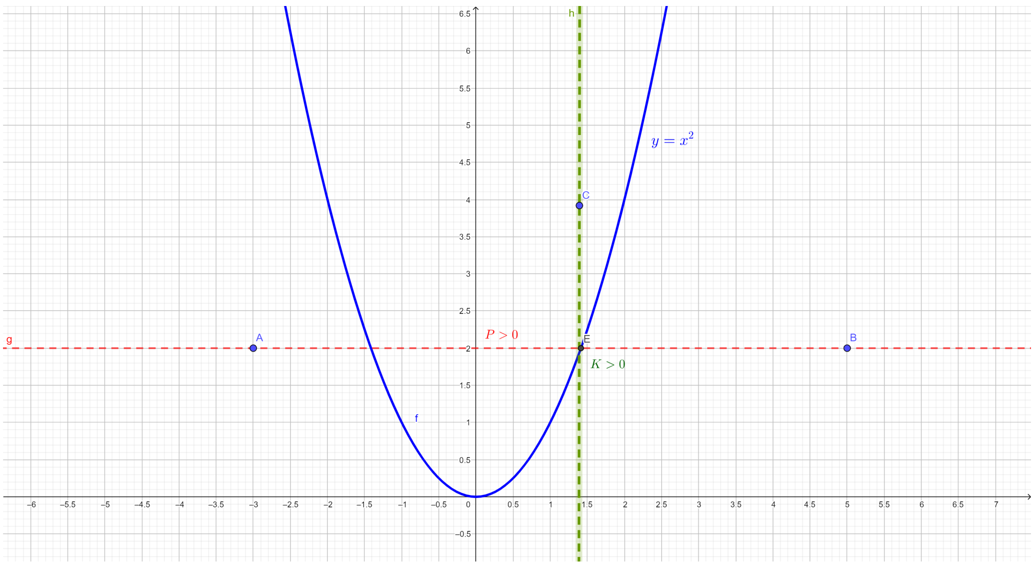
   $$x > \sqrt{P} \text{ or } x < -\sqrt{P}.$$

   Since $x > 0$ was assumed, we choose as solution only the $x > \sqrt{P}$ case.

   So $K := \sqrt{P} + 1 > 0$ is a (possible) good lower bound for the definition:

   $$\forall\, P > 0 \;\exists\, K := \sqrt{P} + 1 > 0 \;: \forall\, x \in (K; +\infty) \cap \mathbb{R} = (K; +\infty) \;:\; x^2 > P.$$

   Examine the meaning of the definition on the following picture:



2. Prove by definition that: $\lim\limits_{x\to+\infty} (2x^5 - 7x^4 + 3x^3 + 11x - 37) = +\infty$.

   Solution:

   - We have to prove the following statement:

   $$\forall\, P > 0 \;\exists\, K > 0 \;: \forall\, x \in (K; +\infty) \cap D_f \;:\; 2x^5 - 7x^4 + 3x^3 + 11x - 37 > P.$$

   Fix a number $P > 0$ and check when is:

   $$f(x) = 2x^5 - 7x^4 + 3x^3 + 11x - 37 > P.$$

In this case we cannot solve it, so the usual technic is to use the OPL estimations. Assume $x \geq 1$ and:

$$f(x) = 2x^5 - 7x^4 + 3x^3 + 11x - 37 \geq 2x^5 - 7x^4 - 37 = 2x^5 - (7x^4 + 37) \geq (\text{for } x \geq 1) \geq$$

$$\geq 2x^5 - 44x^4 = x^5 + x^5 - 44x^4 = x^5 + x^4 \cdot (x - 44) \geq (\text{if } x \geq 44) \geq x^5 > (?) > P.$$

Now its easy to solve:

$$x^5 > P > 0 \iff x > \sqrt[5]{P}.$$

Since we need $x \geq 44$ as well, then $K := \sqrt[5]{P} + 44 > 0$ will be good for the definition.

$$\forall\, P > 0 \;\exists\, K := \sqrt[5]{P} + 44 > 0 \;:\, \forall\, x \in (K; +\infty) \cap \mathbb{R} = (K; +\infty) \;:$$

$$2x^5 - 7x^4 + 3x^3 + 11x - 37 > P.$$

3. Prove by definition that: $\displaystyle\lim_{x \to +\infty} \frac{3x^4 + x^2 - 3x + 5}{2x^3 + 2x + 1} = +\infty.$

Solution:

• Let's see first what does this limit mean? Consider the given polynomials in the numerator and denominator, and assume that $x > 0$:

$$P(x) := 3x^4 + x^2 - 3x + 5 = x^4 \cdot (3 + 1/x^2 - 3/x^3 + 5/x^4) \to (+\infty)^4 \cdot (3 + 0 - 0 + 0) =$$

$$= (+\infty) \cdot 3 = +\infty \quad (\text{as } x \to +\infty).$$

So as x is a great number, then $P(x)$ is also great, or as x gets close to $+\infty$, then the values $P(x)$ also gets close to $+\infty$. In the same manner:

$$Q(x) := 2x^3 + 2x + 1 = x^3 \cdot (2 + 2/x + 1/x^3) \to (+\infty)^3 \cdot (2 + 0 + 0) = (+\infty) \cdot 2 = +\infty,$$

as $x \to +\infty$. Here we also can say that as $x \approx +\infty$ implies $Q(x) \approx +\infty$.

But what happens with our fraction as $x \approx +\infty$?

$$f(x) = \frac{P(x)}{Q(x)} = \frac{3x^4 + x^2 - 3x + 5}{2x^3 + 2x + 1} \approx \frac{+\infty}{+\infty}.$$

This last approximation cannot be defined, we will call it a critical limit, and whenever this case is on, we factor out the *dominant terms*, so:

$$\lim_{x \to +\infty} \frac{3x^4 + x^2 - 3x + 5}{2x^3 + 2x + 1} = \lim_{x \to +\infty} \frac{x^4 \cdot (3 + 1/x^2 - 3/x^3 + 5/x^4)}{x^3 \cdot (2 + 2/x^2 + 1/x^3)} =$$

$$= \lim_{x \to +\infty} (x) \cdot \lim_{x \to +\infty} \frac{3 + 1/x^2 - 3/x^3 + 5/x^4}{2 + 2/x^2 + 1/x^3} = (+\infty) \cdot \frac{3}{2} = +\infty.$$

This process will be the way to evaluate limits of rational functions at $+\infty$, using the corresponding theorems of calculus. Now the limit is given, and we have to prove it by definition.

- What do we have to prove here by definition? The following statement:

$$\forall\, P>0\ \exists\, K>0\, \forall\, x\in (K;+\infty)\cap D_f\ :\ \frac{3x^4+x^2-3x+5}{2x^3+2x+1}>P.$$

We start by fixing a $P>0$ number. (This is the error to approximate the limit $+\infty$, so how "close" do we want to get with $f(x)$ to infinity: we want to fulfill $f(x)>P$, when x is going to be great enough.) Suppose now that $x>0$. (This is a so called "initial restriction". The domain D_f is the set of all real numbers, except the zeros of the denominator. Many times we cannot evaluate them, but no problem, we will choose a positive K great enough number, so that when $x>K$, then the denominator will also not be zero. In our case $Q(x)=2x^3+2x+1>0$, when $x>0$ is true.) We continue by the usual OPL estimations ($f(x)\geq \ldots \geq C\cdot \frac{x^4}{x^3}=C\cdot x$):

$$f(x)=\frac{3x^4+x^2-3x+5}{2x^3+2x+1}\geq (\text{if } x\geq 1)\geq \frac{3x^4-3x}{2x^3+2x+1}\geq (\text{if } x\geq 1)\geq$$

$$\geq \frac{2x^4+x\cdot(x^3-3)}{2x^3+2x^3+x^3}\geq (\text{if } x\geq 2)\geq \frac{2x^4}{5x^3}=\frac{2}{5}x>(?)>P.$$

- At the end, now we can answer when is

$$\frac{2}{5}x>P \iff x>\frac{5P}{2}.$$

So if we choose $K:=\frac{5P}{2}+2>0$, then the statement is fulfilled.

4. Prove by definition that: $\lim\limits_{x\to+\infty}\sqrt{x-11}=+\infty$.

Solution:

- We have to prove the following statement:

$$\forall\, P>0\ \exists\, K>0\ :\forall\, x\in (K;+\infty)\cap D_f=[11;+\infty)\ :\ \sqrt{x-11}>P.$$

Fix a $P>0$ number. Assume $x\geq 11$ and:

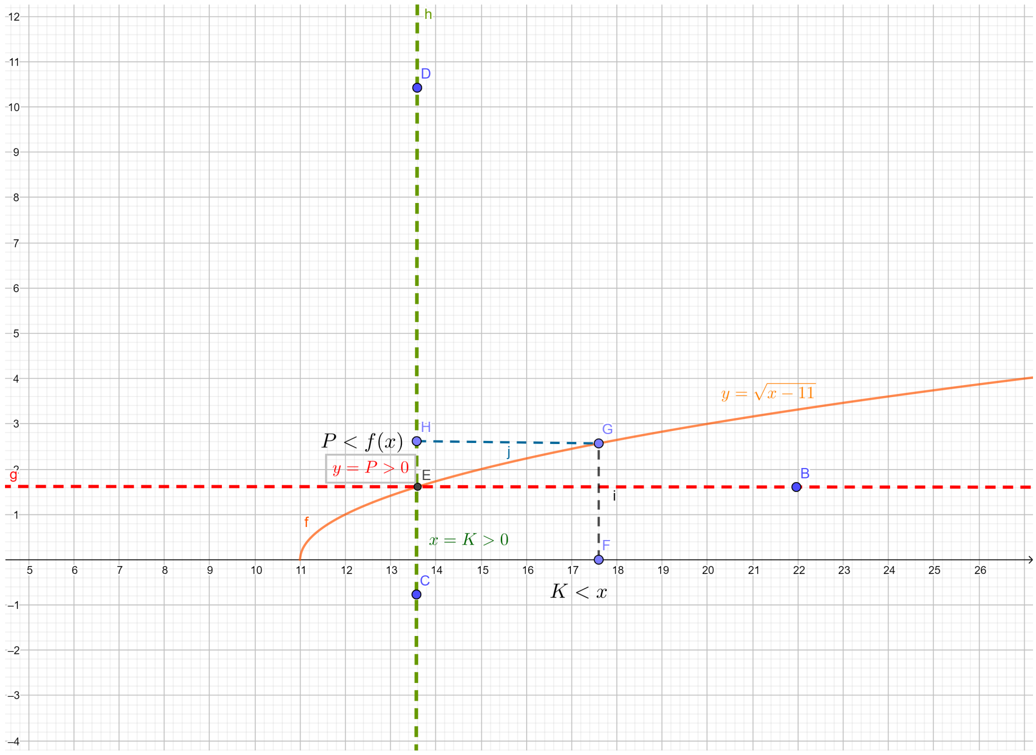
$$f(x)=\sqrt{x-11}>P>0 \iff x-11>P^2 \iff x>P^2+11.$$

We can conclude, that $K:=P^2+11>0$ will be good for the definition:

$$\forall\, P>0\ \exists\, K:=P^2+11>0\ :\forall\, x\in (K;+\infty)\cap[11;+\infty)=(P^2+11;+\infty):$$

$$\sqrt{x-11}>P.$$

On the following image one can visually see the given proof by definition:



Def: *(Finite limit at $+\infty$)* Assume $f \in \mathbb{R} \longrightarrow \mathbb{R}$ is a real-to -real function again and $+\infty$ is an accumulation point of D_f. We say that f has a finite limit $L \in \mathbb{R}$ at $+\infty$, in notation

$$\lim_{x\to+\infty} f(x) = L,$$

if the following logical statement is true:

$$\forall\, \varepsilon > 0 \;\; \exists\, K > 0 \,\forall\, x \in (K; +\infty) \cap D_f \;:\; |f(x) - L| < \varepsilon.$$

1. Prove by definition, that: $\displaystyle\lim_{x\to+\infty} \frac{x^2 - x + 1}{2x^2 + 5x + 31} = \frac{1}{2}.$

   Solution:

   - We need to prove the following statement:

$$\forall\, \varepsilon > 0 \;\; \exists\, K > 0 \,\forall\, x \in (K; +\infty) \cap D_f : \left| \frac{x^2 - x + 1}{2x^2 + 5x + 31} - \frac{1}{2} \right| < \varepsilon.$$

   where $D_f = \{x \in \mathbb{R} \mid 2x^2 + 5x + 31 \neq 0\} = \mathbb{R}$.

   Fix an $\varepsilon > 0$ number, and evaluate $|f(x) - L|$:

$$|f(x) - 1/2| = \left| \frac{x^2 + 4x + 1}{2x^2 + 5x + 31} - \frac{1}{2} \right| = \left| \frac{2x^2 + 8x + 2 - (2x^2 + 5x + 31)}{4x^2 + 10x + 62} \right| =$$

$$= \left|\frac{3x-29}{4x^2+10x+62}\right| = \frac{|3x-29|}{|4x^2+10x+62|} \leq (\triangle) \leq \frac{|3x|+|-29|}{4x^2+10x+62} = \frac{3|x|+29}{4x^2+10x+62}.$$

Assume, that $x \geq 1$:

$$|f(x)-1/2| \leq \frac{3|x|+29}{4x^2+10x+62} = \frac{3x+29}{4x^2+10x+62} \leq \frac{3x+29x}{4x^2} = \frac{32x}{4x^2} = \frac{8}{x} < \varepsilon \iff$$

$$\iff x > \frac{8}{\varepsilon}.$$

So if we choose $K := \dfrac{8}{\varepsilon} + 1 > 0$ and $x > K$ then our definition is fulfilled.

2. Prove by definition, that:

$$\lim_{x\to+\infty} \frac{-2x^3-15x^2+x-12}{5x^3+x^2+9x+1} = -\frac{2}{5}.$$

Solution:

- We need to prove the following statement:

$$\forall\, \varepsilon > 0\ \exists\, K > 0\ : \forall\, x \in (K;+\infty) \cap D_f : \left|\frac{-2x^3-15x^2+x-12}{5x^3+x^2+9x+1} + \frac{2}{5}\right| < \varepsilon.$$

where $D_f = \{x \in \mathbb{R} \mid 5x^3+x^2+9x+1 \neq 0\}$.

Fix an $\varepsilon > 0$ number, assume $x \geq 1$, and evaluate $|f(x) - L|$:

$$|f(x) - (-2/5)| = \left|\frac{-2x^3-15x^2+x-12}{5x^3+x^2+9x+1} + \frac{2}{5}\right| =$$

$$= \left|\frac{-10x^3-75x^2+5x-60+10x^3+2x^2+18x+2}{5\cdot(5x^3+x^2+9x+1)}\right| = \left|\frac{-73x^2+23x-58}{5\cdot(5x^3+x^2+9x+1)}\right| \leq (\triangle) \leq$$

$$\leq \frac{|-73x^2|+|23x|+|-58|}{25x^3+5x^2+45x+5} = \frac{73x^2+23|x|+58}{25x^3+5x^2+45x+5} = \frac{73x^2+23x+58}{25x^3+5x^2+45x+5} \leq$$

$$\leq (\text{if } x \geq 1) \leq \frac{73x^2+23x^2+58x^2}{25x^3} = \frac{154}{25x} < \varepsilon \iff x > \frac{154}{25\varepsilon}.$$

So if we choose $K := \dfrac{154}{25\varepsilon} + 1 > 0$, then our definition is fulfilled.

Follow the steps of this proof on this picture:

3. Prove by definition, that:

$$\lim_{x\to+\infty}(\sqrt{x+4}-\sqrt{x+1})=0.$$

Solution:

- We need to prove the following statement:

$$\forall\,\varepsilon>0\ \exists\,K>0\ :\forall\,x\in(K;+\infty)\cap D_f:\left|(\sqrt{x+4}-\sqrt{x+1})-0\right|<\varepsilon.$$

where $D_f=[-1;+\infty)$.

Fix an $\varepsilon>0$ number, assume $x\geq 1$, and evaluate $|f(x)-L|$:

$$|f(x)-0|=\left|\sqrt{x+4}-\sqrt{x+1}\right|=\sqrt{x+4}-\sqrt{x+1}=$$

$$=\frac{(\sqrt{x+4}-\sqrt{x+1})\cdot(\sqrt{x+4}+\sqrt{x+1})}{\sqrt{x+4}+\sqrt{x+1}}=\frac{(x+4)-(x+1)}{(\sqrt{x+4}+\sqrt{x+1})}=\frac{3}{\sqrt{x+4}+\sqrt{x+1}}.$$

When is the last expression small enough ($<\varepsilon$)?

$$|f(x)-0|=\frac{3}{\sqrt{x+4}+\sqrt{x+1}}\leq\frac{3}{\sqrt{x+1}}\leq\ (\text{if } x\geq 1)\ \leq\frac{3}{\sqrt{x}}<\varepsilon\quad\iff$$

$$\iff\quad\sqrt{x}>\frac{3}{\varepsilon}>0\quad\iff\quad x>\frac{9}{\varepsilon^2}.$$

So if we choose $K:=\dfrac{9}{\varepsilon^2}+1>0$, then our definition is fulfilled.

4. Prove by definition, that:
$$\lim_{x\to+\infty}\frac{\sin(2x)}{x}=0.$$

Solution:

- We need to prove the following statement:

$$\forall\,\varepsilon>0\ \exists\,K>0\ :\forall\,x\in(K;+\infty)\cap D_f:\left|\frac{\sin(2x)}{x}-0\right|<\varepsilon.$$

where $D_f=\mathbb{R}\setminus\{0\}$.

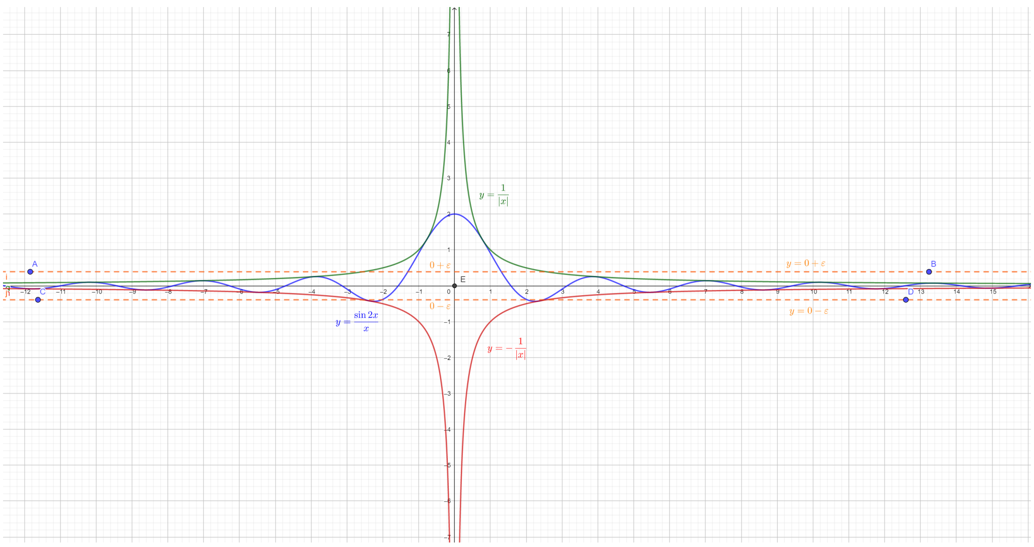
Fix an $\varepsilon>0$ number, assume $x\geq 1$, and evaluate $|f(x)-L|$:

$$|f(x)-0|=\left|\frac{\sin(2x)}{x}\right|=\frac{|\sin(2x)|}{|x|}\leq\frac{1}{|x|}=(\text{if } x\geq 1)=\frac{1}{x}<\varepsilon\iff x>\frac{1}{\varepsilon}.$$

So, we have:

$$\forall\,\varepsilon>0\ \exists\,K:=\frac{1}{\varepsilon}+1>0\ :\forall\,x\in(K;+\infty)\cap\mathbb{R}\setminus\{0\}=(K;+\infty):$$
$$\left|\frac{\sin(2x)}{x}-0\right|\leq\frac{1}{x}<\frac{1}{K}<\varepsilon.$$

Follow the steps of this proof on this picture:



Def: *(Negative infinity as limit at $+\infty$)* Suppose $f\in\mathbb{R}\longrightarrow\mathbb{R}$ is a real-to-real function with $+\infty\in D'_f$. We state that f has limit $-\infty$ at $+\infty$, in notation

$$\lim_{x\to+\infty}f(x)=-\infty,$$

when the following statement is true:

$$\forall\,p<0\ \exists\,K>0\ :\forall\,x\in(K;+\infty)\cap D_f\ :\ f(x)<p.$$

1. Prove by definition that: $\lim\limits_{x\to+\infty}(2-\sqrt{x-5}) = -\infty$.

   Solution:

   - We have to prove the following statement:

   $$\forall\, p<0\ \ \exists\, K>0\, \forall\, x \in (K;+\infty)\cap[5;+\infty)\ :\ 2-\sqrt{x-5}<p.$$

   Fix a number $p<0$, and make sure, that for all great enough x values:

   $$f(x)=2-\sqrt{5-x}<p\,|\cdot(-1)\quad\Longleftrightarrow\quad -f(x)=\sqrt{x-5}-2>-p\ (>0).$$

   Now we can use the technik seen for the $+\infty$ case, so:

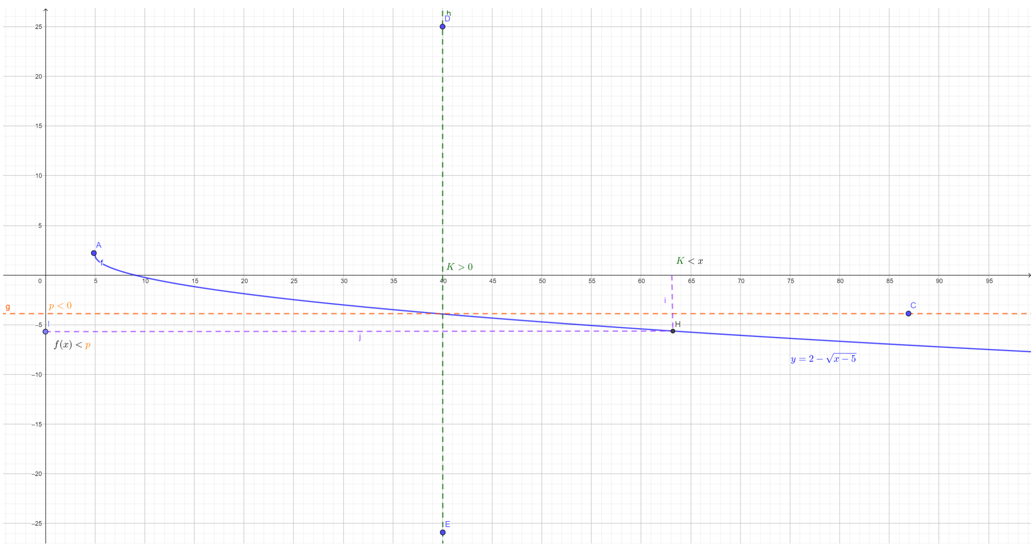
   $$-f(x)=\sqrt{x-5}-2\geq -p\quad\Longleftrightarrow\quad \sqrt{x-5}>2-p>0\ ekvi\ x>(2-p)^2+5.$$

   Since $x\geq 5$ is the domain, we can choose $K:=(2-p)^2+6>0$ to be good for the definition, so:

   $$\forall\, p<0\ \ \exists\, K:=(2-p)^2+6>0\ :\forall\, x\in(K;+\infty)\cap[5;+\infty)=(K;+\infty)\ :$$

   $$\sqrt{x-5}-2>-p\quad\Longleftrightarrow\quad 2-\sqrt{x-5}<p.$$

   Check this proof on the following picture:



2. Prove by definition that: $\lim\limits_{x\to+\infty}(-2x^3+x^2+28x-111) = -\infty$.

   Solution:

   - We have to prove the following statement:

   $$\forall\, p<0\ \ \exists\, K>0\, \forall\, x\in(K;+\infty)\cap D_f=\mathbb{R}\ :\ -2x^3+x^2+28x-111<p.$$

Fix a $p<0$ number, and make sure, that for all great enough x values:

$$f(x) = -2x^3+x^2+28x-111 < p \;|\cdot(-1) \iff -f(x) = 2x^3-x^2-28x+111 > -p \;(>0).$$

Now we can use the technik seen for the $+\infty$ case, so:

$$-f(x) = 2x^3-x^2-28x+111 \geq 2x^3-(x^2+29x) \geq (\text{for } x\geq 1) \geq 2x^3-29x^2 =$$

$$= x^3+x^3-29x^2 = x^3+x^2\cdot(x-29) \geq (\text{if } x\geq 29) \geq x^3 > (?) > -p.$$

Solving this last inequality:

$$x^3 > -p > 0 \iff x > \sqrt[3]{-p}.$$

Since we need $x \geq 29$ as well, then $K := \sqrt[3]{-p}+29 > 0$ will be good for the definition, so:

$$\forall\, p<0 \;\exists\, K := \sqrt[3]{-p}+29>0 \;: \forall\, x \in (K;+\infty)\cap\mathbb{R} = (K;+\infty) \;:$$

$$2x^3-x^2-28x+111 > -p \iff -2x^3+x^2+28x-111 < p.$$

3. Prove by definition, that: $\lim\limits_{x\to+\infty} \dfrac{-2x^7+3x^5-x^3+2x^4+x+5}{3x^5+4x^2+x+3} = -\infty.$

Solution:

- We need to prove the following statement:

$$\forall\, p<0 \;\exists\, K>0\,\forall\, x\in(K;+\infty)\cap D_f : \frac{-2x^7+3x^5-x^3+2x^4+x+5}{3x^5+4x^2+x+3} < p,$$

where $D_f = \{x\in\mathbb{R} \mid 3x^5+4x^2+x+3 \neq 0\}$.

Fix a $p<0$ number. (This is the error to approximate the limit $-\infty$, so how "close" do we want to get with values $f(x)$ to $-\infty$. We want to assure $f(x)<p$, when x is great enough.)

$$\frac{-2x^7+3x^5-x^3+2x^4+x+5}{3x^5+4x^2+x+3} < p \;|\cdot(-1) \iff$$

$$\iff \frac{+2x^7-3x^5+x^3-2x^4-x-5}{3x^5+4x^2+x+3} > -p =: P > 0.$$

Assume, that $x>0$. In this case $Q(x) = 3x^5+4x^2+x+3>0$ is also true (so the denominator is not zero). We do the usual OPL estimations (for this all the main coefficients must be positive numbers).

$$-f(x) \geq \ldots \geq C\cdot\frac{2x^7}{3x^5} = C\cdot x^2.$$

$$-f(x) = \frac{2x^7-3x^5+x^3-2x^4-x-5}{x^5+4x^2+x+3} \geq (\text{if } x\geq 1) \geq \frac{2x^7-(3x^5+2x^4+x+5)}{3x^5+4x^2+x+3} \geq$$

$$\geq (\text{if } x \geq 1) \geq \frac{2x^7 - 11x^5}{3x^5 + 4x^5 + x^5 + 3x^5} = \frac{x^7 + x^7 - 11x^5}{11x^5} =$$

$$= \frac{x^7 + x^5 \cdot (x^2 - 11)}{11x^5} \geq (\text{if } x \geq 4) \geq \frac{x^7}{11x^5} = \frac{1}{11}x^2 > (?) > -p.$$

- Solve now:

$$\frac{1}{11} \cdot x^2 > -p \iff x^2 > -11p > 0 \iff (x < -\sqrt{-11p}) \text{ or } (x > \sqrt{-11p} > 0).$$

So if we choose $K := \sqrt{-11p} + 4 > 0$, then our definition is fulfilled.

3.1.1. Checking questions to the theory and its use

1. When do we say that a real-to-real function has a *maximal value*?
2. When do we say that a real-to-real function has a *minimal value*?
3. When do we say that a real-to-real function is *bounded from below*?
4. When do we say that a real-to-real function is *bounded from above*?
5. When do we say that a real-to-real function is *bounded*?
6. Give the logical statement for the definition to a function not to be *bounded from above*.
7. Give the definition of the $+\infty$ limit of a function at $+\infty$.
8. Give the defintion of $\lim\limits_{x\to+\infty} f(x) = 2$.
9. Give the definition of $\lim\limits_{x\to+\infty} f(x) = -\infty$.
10. Is there any function $f : \mathbb{R} \longrightarrow \mathbb{R}$, which has minimal and maximal values at every point of its domain?
11. Give an example of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which has a lower bound, but it is not bounded from above.
12. Give an example of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which has a lower bound, it has no minimal value and it is not bounded from above.
13. Give an example of a function $f : (0; 1) \longrightarrow \mathbb{R}$ which is not bounded below, but it is bounded from above.
14. Is there a minimal value and place for $f(x) := x^2 + \left(\frac{1}{x}\right)^2 \quad (x \in \mathbb{R} \setminus \{0\})$?
15. Prove by definition that $\lim\limits_{x\to+\infty} \frac{x^4 + 1}{x^2} = +\infty$.

16. Prove by definition that $\lim\limits_{x\to+\infty} \dfrac{\sqrt{x}+1}{\sqrt{x}} = 1$.

17. Prove that the function $f(x) := \dfrac{x^4+1}{x^2}$ $(x \in \mathbb{R} \setminus \{0\})$ is not bounded from above.

18. Give an example of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which has no upper and lower bounds.

19. Is it true: when a function is bounded from above it also has a maximal value? If not, give an example.

20. Is it true: when a function has a minimal value then it is also bounded below?

21. Does $\lim\limits_{x\to+\infty} \sin x$ exist?

22. Does $\lim\limits_{x\to+\infty} \dfrac{\sin x}{x}$ exist?

3.2. Exercises

3.2.1. Exercises for class work

Bounded functions, minimal and maximal values

1. Find the greatest and the least values of the following functions:

   (a) $f(x) := x^2 - 4x + 3 \quad (x \in \mathbb{R})$;

   (b) $f(x) := x^2 - 4x + 3 \quad \left(\dfrac{1}{2} \leq x \leq 3\right)$.

2. Consider $a, b \in \mathbb{R}$ and $P(x) := x^2 + ax + b \quad (x \in \mathbb{R})$. Find a, b so that

$$P(3) = \min\{P(x) : x \in \mathbb{R}\} = -2 \ .$$

3. With a 24 meters long fence we would like to fence in (enclose) a rectangular garden so that at one side of this area is the wall of the house (so no fence needed along this side). Of what sizes should be the sides of this rectangle if we want the enclosed area to be of maximal value.

4. Find the area of the largest rectangle that can be inscribed in a right trianle with legs of lengths 10 cm and 15 cm if two sides of the rectangle lie along the legs of the triangle.

5. Examine the following functions regarding boundedness, and mnimal/maximal values:

   (a) $f(x) := \dfrac{3x^2+7}{9x^2+3} \quad (x \in [1;+\infty))$;

   (b) $f(n) := \dfrac{10n+7}{15n+12} \quad (n \in \mathbb{N})$;

   (c) $f(x) := \dfrac{1-3\sqrt{x}}{1+3\sqrt{x}} \quad (x \in [4;+\infty))$;

   (d) $x(n) := x_n := \dfrac{2^{n+1}+2}{3 \cdot 2^n + 1} \quad (n \in \mathbb{N})$;

   (e) $f(x) := \dfrac{x^2+2}{\sqrt{x^2+1}} \quad (x \in \mathbb{R})$;

   (f) $f(x) := \dfrac{\sin^2 x - \sin x \cos x + \cos^2 x}{\sin^2 x + \sin x \cos x + \cos^2 x} \quad (x \in (0;\pi/2))$;

   (g) $f(x) = \sqrt{3} \cdot \sin x + \cos x \quad (x \in \mathbb{R})$.

6. Find the greatest and the least values of the function

$$f(x) := \frac{x^4+1}{x^2+1} \quad (x \in \mathbb{R})$$

   and also give the points where f takes these extermal values.

$\boxed{\text{Estimates of functions, limits at } +\infty}$

7. Consider the function $f(x) := \sqrt{x} \;\; (x \in [0;+\infty))$.

   a) Find a number $K > 0$ so that for all numbers $x > K$ from the domain of definition of f we have:

$$f(x) > 1000 \; .$$

   b) Is the following statement true:

$$\forall\, P > 0 \;\; \exists\, K > 0 \;\; \forall x \in (K;+\infty) \cap D_f : \;\; f(x) > P \;?$$

   c) Give the negation of the upper statement.

   d) Is it true that

$$\lim_{x \to +\infty} \sqrt{x} = +\infty?$$

8. Consider the function $f(x) := 1 - x^2 \;\; (x \in \mathbb{R})$.

   a) Give a number $K > 0$ so that for all numbers $x > K$ from D_f to have:

$$f(x) < -999 \; .$$

b) Is the following statement true:

$$\forall\, p < 0 \;\; \exists\, K > 0 \;\; \forall x \in (K; +\infty) \cap D_f : \;\; f(x) < p\,?$$

c) Give the negation of the upper statement.

d) Is it true, that

$$\lim_{x \to +\infty} (1 - x^2) = -\infty?$$

9. Let $f(x) := \dfrac{x^4 + 1}{x^2 + 1}$ $(x \in \mathbb{R})$ be a function.

a) Give such a $K > 0$ so that for all $x > K$ from the domain of f to have:

$$f(x) > 1000\,.$$

b) Is the following statement true:

$$\forall\, P > 0 \;\; \exists\, K > 0 \;\; \forall x \in (K; +\infty) \cap D_f : \;\; f(x) > P\,?$$

c) Negate statement from point b).

d) Prove that:

$$\lim_{x \to +\infty} \frac{x^4 + 1}{x^2 + 1} = +\infty.$$

10. Consider the function $f(x) := \dfrac{x^2 + 1}{3x^2 - 1}$ $\left(x \in \mathbb{R} \setminus \{\pm 1/\sqrt{3}\}\right)$.

a) Give such a number $K > 0$ so that for all $x > K$ from D_f to have:

$$\left| f(x) - \frac{1}{3} \right| < \frac{1}{600}\,.$$

b) Is the following statement true:

$$\forall\, \varepsilon > 0 \;\; \exists\, K > 0 \;\; \forall x \in (K; +\infty) \cap D_f : \;\; \left| f(x) - \frac{1}{3} \right| < \varepsilon\,?$$

c) Negate the upper statement.

d) Evaluate by definition

$$\lim_{x \to +\infty} \frac{x^2 + 1}{3x^2 - 1}.$$

11. Consider the function $f(x) := \dfrac{x^3+1}{1-x}$ $(x \in \mathbb{R} \setminus \{1\})$.

    a) Give a number $K > 0$ (if it exists), so that for all $x > K$ from D_f to have:
    $$f(x) < -1000.$$

    b) Is it true that:
    $$\forall\, p < 0\ \exists\, K > 0\ \forall x \in (K; +\infty) \cap D_f : \ \ f(x) < p\,?$$

    c) Negate the statement from point b).

    d) Find
    $$\lim_{x\to+\infty} \frac{x^3+1}{1-x}$$
    using the definition.

12. Prove by definition that: $\displaystyle\lim_{x\to+\infty} \frac{x^4-2x^3+x^2+7}{x^3+x+1} = +\infty.$

13. Prove by definition that: $\displaystyle\lim_{x\to+\infty} \frac{2x^3-x^2+3}{x^3+2x-5} = 2.$

14. Prove by definition that: $\displaystyle\lim_{x\to+\infty} \frac{x^3+x^2-2x-3}{9-4x^2} = -\infty.$

15. Prove by definition that: $\displaystyle\lim_{x\to+\infty} \frac{2\cos x}{x} = 0.$

16. Prove by definition that: $\displaystyle\lim_{x\to+\infty} \sqrt{x^2-x+11} = +\infty.$

17. Prove by definition that: $\displaystyle\lim_{x\to+\infty} (\sqrt{2x+1} - \sqrt{2x-1}) = 0.$

18. Evaluate the following limit, and then prove by definition your statement:
    $$\lim_{x\to+\infty} (\sqrt{x^2+x} - x).$$

3.2.2. Homework and more exercises to practice

Bounded functions, extremal values

1. Find the extremal values of f for:

   (a) $f(x) := x^2 + x - 6 \quad (x \in \mathbb{R})$;

   (b) $f(x) := x^2 + x - 6 \quad (x \in [-1; 3])$;

(c) $f(x) := -2 - 2x - x^2 \quad (x \in \mathbb{R})$;

(d) $f(x) := -2 - 2x - x^2 \quad (x \in (-\infty; 0])$;

(e) $f(x) := -2 - 2x - x^2 \quad (x \in [-1/2; 2])$.

2. Find parameters $a, b \in \mathbb{R}$ so that for the polynomial:

$$P(x) := -x^2 + ax + b \qquad (x \in \mathbb{R})$$

to have:

$$P(-1) = \max\{P(x) \mid x \in \mathbb{R}\} = 3.$$

3. Find the right triangle with maximal area so that the sum of its legs is 12 cm.

4. Consider a 20m long segment AB. Find a point C between A and B so that if we consider two half circles of diameter AC and CB the sum of the areas of these half circles to be minimal.

5. Which cylinder has the maximal volume from all the cylinders inscribed into a given cone?

6. Examine the following functions regarding extemal values and boundedness:

(a) $f(x) := \dfrac{3(x-1)^2 + 7}{9(x-1)^2 + 3} \quad (x \in (-\infty; 2])$;

(b) $f(x) := \dfrac{|x| - 1}{5|x| - 2} \quad (x \in [1; +\infty))$;

(c) $f(x) := \dfrac{x^2 - 4x + 5}{(x-2)^2} \quad (x \in (-\infty; 0])$;

(d) $f(x) := \dfrac{x^2 + 2x + 6}{3x^2 + 6x + 9} \quad (x \in \mathbb{R})$;

(e) $f(x) := \dfrac{2x^2 + 6x + 6}{x^2 + 4x + 5} \quad (x \in \mathbb{R})$;

(f) $f(x) := \dfrac{2x - 4\sqrt{x} - 1}{5x - 10\sqrt{x} + 10} \quad (x \in [0; +\infty))$;

(g) $f(x) := x + \dfrac{1}{x} - \sqrt{x^2 + \dfrac{1}{x^2}} \quad (x \in (0; +\infty))$;

(h) $f(x) := \dfrac{7 - 3\sin^2 x + 6\cos x}{\cos^2 x + 2\cos x + 2} \quad (x \in \mathbb{R})$;

(i) $f(n) := x_n := \dfrac{8n + 3}{5n + 4} \quad (n \in \mathbb{N})$;

(j) $f(x) := \dfrac{1 - e^{x+1}}{e^x + 1} \quad (x \in \mathbb{R})$;

(k) $f(x) := \sqrt{x+1} - \sqrt{x} \quad (x \in [0; +\infty))$;

(l) $f(x) = \sin x \cdot (\cos x + \sqrt{3} \cdot \sin x) + \cos x \cdot (\sin x - \sqrt{3} \cos x) \quad (x \in \mathbb{R})$.

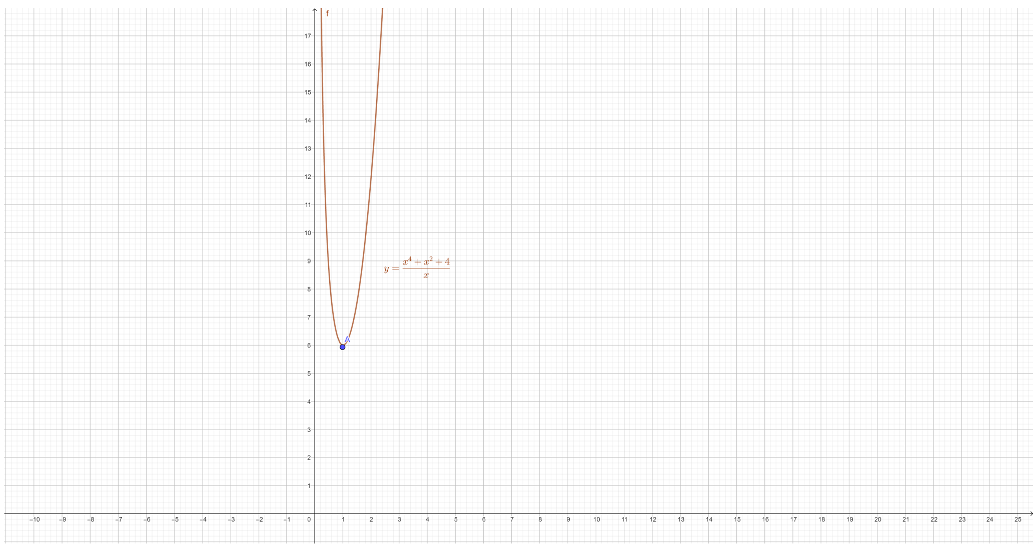
7. Where does the function $f(x) := \dfrac{x^4 + x^2 + 4}{x}$ $(x \in (0; +\infty))$ take its minimal value and what is this value?

Solution: Using the inequality between the arithmetic and geometric means we get:

$$f(x) = x^3 + x + \frac{1}{x} + \frac{1}{x} + \frac{1}{x} + \frac{1}{x} \geq 6 \cdot \sqrt{x^3 \cdot x \cdot \left(\frac{1}{x}\right)^4} = 6.$$

We have minimal value $y = 6$ at $x = 1$.

Here is the graph of f:



8. What are the extremal values of the function $f(x) := \dfrac{x^2}{1 + x^4}$ $(x \in \mathbb{R})$ and where does f take them?

$\boxed{\text{Estimates, limits at } +\infty}$

9. For the function $f(x) := \sqrt{x^2 + 1}$ $(x \in \mathbb{R})$:

a) Find a number $K > 0$ so that for all $x > K$ from the domain D_f we have:

$$f(x) > 100.$$

b) Is the following statement true:

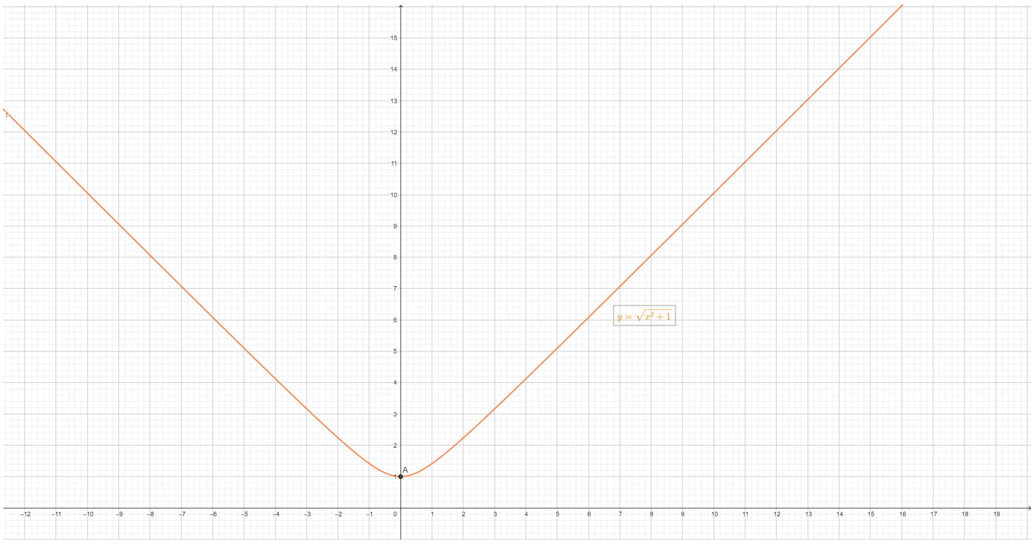
$$\forall\, P > 0 \;\exists\, K > 0 \;\forall x \in (K; +\infty) \cap D_f : \quad f(x) > P\ ?$$

c) Negate the upper statement.

d) Evaluate by definition

$$\lim_{x \to +\infty} \sqrt{x^2 + 1}.$$

Here is the graph of f:



10. Consider the function $f(x) := \ln \frac{1}{x}$ $\ (x \in (0; +\infty))$.

a) Find a number $K > 0$ so that for all $x > K$ from the domain D_f we have:

$$f(x) < -100.$$

b) Is it true that:

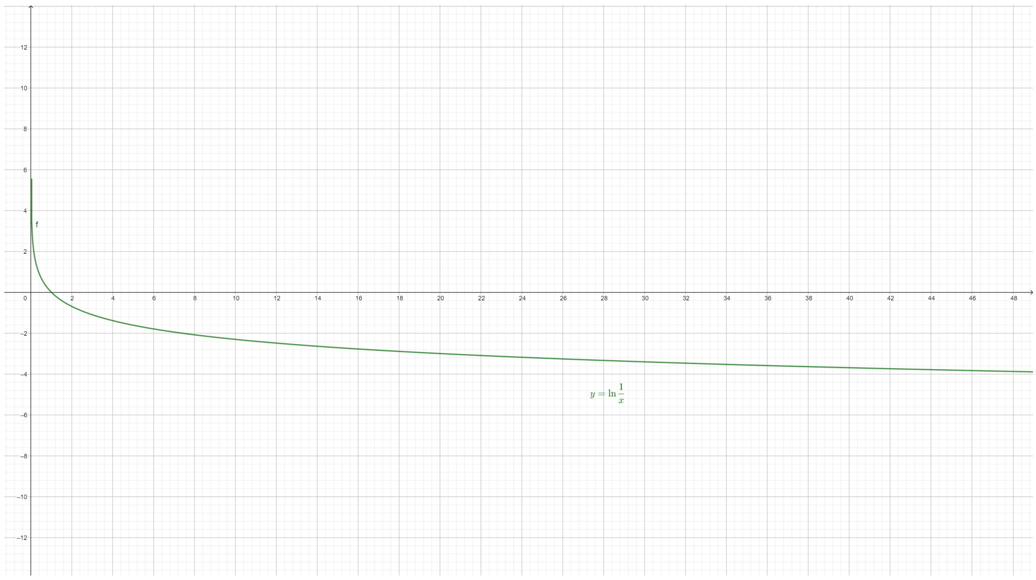
$$\forall\, p < 0 \;\exists\, K > 0 \;\forall x \in (K; +\infty) \cap D_f : \quad f(x) < p\ ?$$

c) Give the negation of the statement in point b).

d) Evaluate by definition

$$\lim_{x \to +\infty} \ln \frac{1}{x}.$$

Here is the graph of f:



11. Consider the function $f(x) := \dfrac{x^6+1}{x^3+x^2}$ $(x \in \mathbb{R} \setminus \{0; -1\})$.

a) Find a number $K > 0$ so that for all $x > K$ from the domain D_f we have:

$$f(x) > 4000.$$

b) Is it true that:

$$\forall\, P > 0 \;\; \exists\, K > 0 \;\; \forall x \in (K; +\infty) \cap D_f : \;\; f(x) > P\ ?$$

c) Negate the upper statement.

d) Evaluate by definition

$$\lim_{x \to +\infty} \frac{x^6+1}{x^3+x^2}.$$

12. Consider the function $f(x) := \dfrac{x^3+x}{1-x^3}$ $(x \in \mathbb{R} \setminus \{1\})$.

a) Find a number $K > 0$ so that for all $x > K$ from the domain D_f we have:

$$|f(x)+1| < \frac{1}{100}.$$

b) Is it true that:

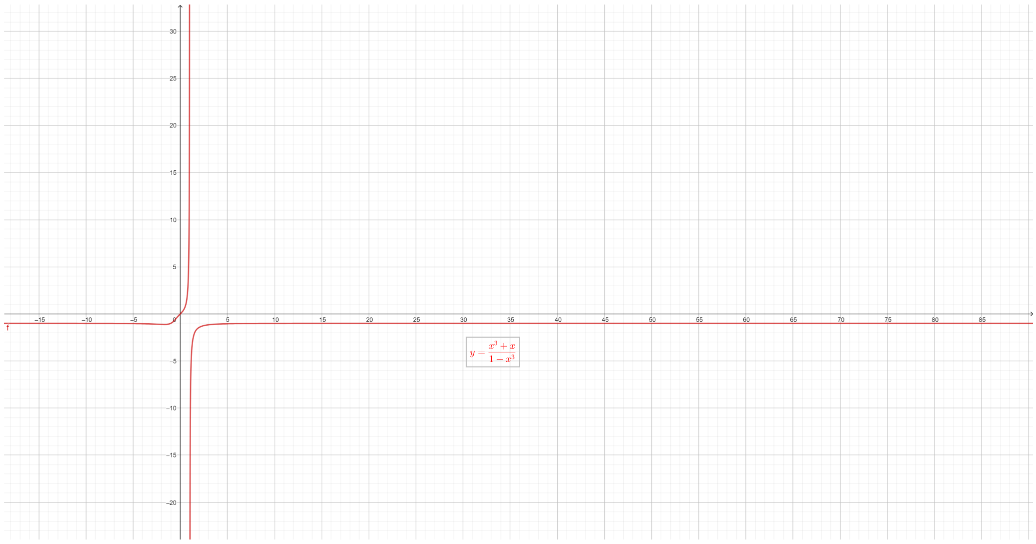
$$\forall\, \varepsilon > 0 \;\; \exists\, K > 0 \;\; \forall x \in (K; +\infty) \cap D_f : \;\; |f(x) - (-1)| < \varepsilon\ ?$$

c) Negate the upper statement.

d) Evaluate by definition

$$\lim_{x \to +\infty} \frac{x^3+x}{1-x^3}.$$

Here is the graph of f:



13. Consider the function $f(x) := \dfrac{1 + x - 3x^5}{x^4 + 16}$ $(x \in \mathbb{R})$.

a) Find a number $K > 0$ so that for all $x > K$ from the domain D_f we have:

$$f(x) < -100.$$

b) Is it true that:

$$\forall\, p < 0 \;\; \exists\, K > 0 \;\; \forall x \in (K; +\infty) \cap D_f : \quad f(x) < p\,?$$

c) Negate the upper statement.

d) Evaluate by definition

$$\lim_{x \to +\infty} \frac{1 + x - 3x^5}{x^4 + 16}.$$

Here is the graph of f:

14. Prove by definition that: $\lim\limits_{x\to+\infty} \dfrac{3x^6 + 2x^5 - 30x^4 + 12x - 7}{x^4 + x^2 - 2x + 10} = +\infty$.

15. Prove by definition that: $\lim\limits_{x\to+\infty} \dfrac{2x^3 + x^2 - 3x + 5}{x^3 + 7x^2 + 2x + 19} = 2$.

16. Prove by definition that: $\lim\limits_{x\to+\infty} \dfrac{x^4 - 21x^3 + x^2 - x - 8}{x^2 - 3x^4 + x + 1} = -\dfrac{1}{3}$.

17. Prove by definition that: $\lim\limits_{x\to+\infty} \dfrac{-2x^3 + 2021x - 2022}{x^2 + 3x + 11} = -\infty$.

18. Prove by definition that: $\lim\limits_{x\to+\infty} \sqrt{3x^3 + x^2 - x + 11} = +\infty$.

19. Prove by definition that: $\lim\limits_{x\to+\infty} \dfrac{2\sqrt{x} - 1}{3\sqrt{x} + 12} = \dfrac{2}{3}$.

20. Evaluate the following limit, and then prove by definition your statement:

$$\lim_{x\to+\infty} (\sqrt{x^2 + 3x + 10} - \sqrt{x^2 + 1}).$$