Analysis-2 lecture schemes
(with Homeworks)[1]

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Contents

1. Lesson 1

1.1. Continuity of functions

Review: The neighbourhood (or ball) of the point $x_0 \in \mathbb{R}$ with radius $r > 0$ is the set

$$B(x_0, r) := \{x \in \mathbb{R} \mid \ |x - x_0| < r\} = (x_0 - r,\, x_0 + r)\,.$$

Using this concept we can define the continuity.

1.1. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in D_f$. f is continuous at "a" $\overset{\text{df}}{\Leftrightarrow}$
$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x \in B(a, \delta) \cap D_f : \quad f(x) \in B(f(a), \varepsilon).$
Let us denote the set of functions that are continuous at "a" by $C(a)$.

From the definition it follows immediately that

- if "a" is an isolated point of D_f then f is continuous at "a".
- if "a" is an accumulation point of D_f then

$$f \quad \text{is continuous at "}a\text{"} \quad \Leftrightarrow \quad \lim_{x \to a} f(x) = f(a).$$

1.2. Definition Let $f \in \mathbb{R} \to \mathbb{R}$. The function f is called to be continuous if it is continuous at every point of its domain, that is

$$\forall\, a \in D_f : \quad f \in C(a).$$

Using this observation and our knowledge about the limit of functions, we can state that the following functions are continuous at every point of their domain, so they are continuous functions:

Constant function, identity function, polynomials, rational functions (e. g.: $1/x$), analytical functions (e. g.: $\exp,\ \sin,\ \cos,\ \mathrm{sh}\ ,\ \mathrm{ch}$).

1.3. Theorem *[the Transference Theorem for continuity] Using our notations:*

$$f \in C(a) \quad \Leftrightarrow \quad \forall\, x_n \in D_f \quad (n \in \mathbb{N}),\ \lim x_n = a : \quad \lim f(x_n) = f(a).$$

The proof of the Transference Theorem is similar to that of the case of limit.

Using the Transference Theorem it is easy to see that

$$f,\ g \in C(a),\ c \in \mathbb{R} \quad \Rightarrow \quad f + g,\ f - g,\ f \cdot g,\ f/g,\ c \cdot f \in C(a),$$

moreover

$$g \in C(a),\ f \in C(g(a)) \quad \Rightarrow \quad f \circ g \in C(a).$$

1.2. Discontinuities

1.4. Definition Let $f \in \mathbb{R} \to \mathbb{R}$ $a \in D_f$. We say that f has a discontinuity at "a" if $f \notin C(a)$.

Remark that continuity and discontinuity are defined only at the points of the domain and are not defined at points outside of the domain.

The discontinuities of an $\mathbb{R} \to \mathbb{R}$ function are classified in the following way:

1.5. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in D_f$, $f \notin C(a)$. We say that "a" is a point of

- removable discontinuity $\Leftrightarrow \exists \lim\limits_{a} f, \quad$ but $\quad \lim\limits_{a} f \neq f(a)$.
- jump $\Leftrightarrow \exists \lim\limits_{a-} f$ and $\exists \lim\limits_{a+} f, \quad$ but $\quad \lim\limits_{a-} f \neq \lim\limits_{a+} f$.
- discontinuity of second kind $\Leftrightarrow \nexists \lim\limits_{a-} f \quad$ or $\quad \nexists \lim\limits_{a+} f$.

1.6. Examples

1. The function $f(x) = \dfrac{1}{x}$ is a continuous function.
2. The function $f(x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$ has jump at 0.
3. The function $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$ has removable discontinuity at 0.

1.3. Compact sets

Once more recall that the neighbourhood (or ball) of the point $x_0 \in \mathbb{R}$ with radius $r > 0$ is the set

$$B(x_0, r) := \{x \in \mathbb{R} \mid \ |x - x_0| < r\} = (x_0 - r,\, x_0 + r)\,.$$

Via the concept of ball we can define important classes of points in connection of a set.

1.7. Definition Let $\emptyset \neq H \subset \mathbb{R}$, $x_0 \in \mathbb{R}$. Then

1. x_0 is an interior point of H, if $\exists\, r > 0 : \quad B(x_0, r) \subseteq H$.

2. x_0 is an exterior point of H, if $\exists\, r > 0 : \quad B(x_0, r) \cap H = \emptyset$ that is $B(x_0, r) \subseteq \overline{H}$. Here $\overline{H}$ denotes the complement of H that is $\overline{H} = \mathbb{R} \setminus H$.

3. x_0 is a boundary point of H, if

   $\forall\, r > 0 : \quad B(x_0, r) \cap H \neq \emptyset$ and $B(x_0, r) \cap \overline{H} \neq \emptyset$.

1.8. Remark. Every interior point lies in H, every exterior point lies in $\overline{H}$. But a boundary point can belong to H or to its complement.

1.9. Definition 1. The set of the interior points of H is called the interior of H and is denoted by $intH$. So

$$intH := \{x_0 \in \mathbb{R} \mid \exists\, r > 0 : \quad B(x_0, r) \subseteq H\} \subseteq H.$$

2. The set of the exterior points of H is called the exterior of H and is denoted by $extH$. So

$$extH := \{x_0 \in \mathbb{R} \mid \exists\, r > 0 : \quad B(x_0, r) \subseteq \overline{H}\} \subseteq \overline{H}.$$

3. The set of the boundary points of H is called the bound of H and is denoted by ∂H. So

$$\partial H := \{x_0 \in \mathbb{R} \mid \forall\, r > 0 : \quad B(x_0, r) \cap H \neq \emptyset \quad \text{and} \quad B(x_0, r) \cap \overline{H} \neq \emptyset\} \subset \mathbb{R}.$$

1.10. Remark. $\mathbb{R} = intH \cup \partial H \cup extH$ (union of disjoint sets).

1.11. Definition Let $H \subseteq \mathbb{R}$. Then

1. H is called an open set $\overset{\text{df}}{\Leftrightarrow} \partial H \subseteq \overline{H}$.

2. H is called a closed set $\overset{\text{df}}{\Leftrightarrow} \partial H \subseteq H$.

1.12. Remarks. So H is open if and only if it does not contain any boundary point and is closed if and only if it contains all of its boundary points

$\emptyset$ and $\mathbb{R}$ are open and closed sets at the same time. There is no other set in $\mathbb{R}$ that is open and closed at the same time.

H is open $\Leftrightarrow$ $\overline{H}$ is closed, H is closed $\Leftrightarrow$ $\overline{H}$ is open.

H is open $\Leftrightarrow H \subseteq intH \Leftrightarrow H = intH$.

About the characterization of closed sets we present the following theorem without proof:

1.13. Theorem *Let $\emptyset \neq H \subseteq \mathbb{R}$. Then H is closed if and only if*

$$\forall\, x_n \in H \ (n \in \mathbb{N}) \ \textit{convergent sequence}: \ \lim_{n\to\infty} x_n \in H\,.$$

After these preliminaries we can define the concept of compact sets.

1.14. Definition Let $\emptyset \neq H \subseteq \mathbb{R}$. H is called a compact set if

$$\forall\, x_n \in H \;(n \in \mathbb{N}) \text{ sequence } \quad \exists\, (x_{\nu_n}) \text{ subsequence}: \quad (x_{\nu_n}) \text{ is convergent and } \lim_{n\to\infty} x_{\nu_n} \in H\,.$$

The $\emptyset$ is called to be compact by definition.

Remark that from the definition it follows immediately that a compact set is closed.

1.15. Theorem *Let $\emptyset \neq H \subseteq \mathbb{R}$. Then H is compact if and only if it is closed and bounded.*

Proof. First suppose that H is compact. Then H is closed as noted above. Suppose indirectly that H is unbounded. Then

$$\forall\, n \in \mathbb{N}\; \exists\, x_n \in H: \quad |x_n| > n\,.$$

By this way we have defined a sequence $x_n \in H$ $(n \in \mathbb{N})$. Taking a subsequence (x_{ν_n}) we have

$$|x_{\nu_n}| > \nu_n \geq n \quad (n \in \mathbb{N})\,.$$

So (x_{ν_n}) is not bounded which implies that it is not convergent. Therefore (x_n) does not contain convergent subsequence.

Conversely, suppose that H is a closed and bounded set and let $x_n \in H$ $(n \in \mathbb{N})$ be a sequence in H. Then (x_n) is bounded so by the Bolzano-Weierstrass theorem (see: Analysis-1) it has a convergent subsequence (x_{ν_n}). Using that H is closed we have $\lim\limits_{n\to\infty} x_{\nu_n} \in H$. □

1.16. Remarks. The theorem is valid if we take $\mathbb{R}^n$ instead of $\mathbb{R}$ and norm instead of absolute value (see: Analysis-3).

In infinite dimensional normed spaces the theorem is not valid. Every compact set is closed and bounded but there exists a closed and bounded set in the space that is not compact (see: Functional Analysis).

The following theorem is very important from the point of view of the extreme values of functions. Recall that

$\alpha = \min H$ is minimal element of H if $\alpha \in H$ and $\forall\, x \in H: \; x \geq \alpha$.

Respectively:

$\beta = \max H$ is maximal element of H if $\beta \in H$ and $\forall\, x \in H: \; x \leq \beta$.

1.17. Theorem *Let $\emptyset \neq H \subseteq \mathbb{R}$ be a compact set in $\mathbb{R}$. Then H has minimal element* $\min H$ *and maximal element* $\max H$.

Proof. We will prove the case $\max H$, the case of minimum can be proved similarly.

H is compact $\Rightarrow$ H is bounded $\Rightarrow$ H is bounded above $\Rightarrow$ $\exists\, \alpha = \sup H \in \mathbb{R}$. We need to prove that $\alpha \in H$.

To show this use the fact that for every $n \in \mathbb{N}$ the number $\alpha - \dfrac{1}{n}$ is not an upper bound, so

$$\forall\, n \in \mathbb{N}\ \exists\, x_n \in H: \quad x_n > \alpha - \frac{1}{n}\,.$$

α is upper bound so we have

$$\alpha - \frac{1}{n} < x_n \le \alpha\,.$$

Let $n \to \infty$ and use the Sandwich Theorem (see: Analysis-1) to obtain: $\lim\limits_{n\to\infty} x_n = \alpha$.

Since H is closed we have $\alpha = \lim\limits_{n\to\infty} x_n \in H$. □

1.4. Homeworks

Discuss the continuity of the following functions (at which points of the domain is it continuous, at which points is it not, the type of the discontinuities, e.t.c.

1.
$$f(x) := \begin{cases} \dfrac{x-2}{x^2-5x+6} & \text{if } x \in \mathbb{R}\setminus\{2;3\} \\[2ex] 0 & \text{if } \quad x \in \{2;3\} \end{cases}$$

2.
$$f(x) := \begin{cases} \dfrac{(x-2)^2}{x^2-5x+6} & \text{if } x \in \mathbb{R}\setminus\{2;3\} \\[2ex] 0 & \text{if } \quad x \in \{2;3\} \end{cases}$$

3.
$$f(x) := \begin{cases} 1-x^2 & \text{if } \quad x \le 0 \\[2ex] (1-x)^2 & \text{if } \ 0 < x \le 2 \\[2ex] 4-x & \text{if } \quad x > 2 \end{cases}$$

4.
$$f(x) := \begin{cases} \dfrac{3-\sqrt{x}}{9-x} & \text{if } x \ge 0,\ x \ne 9 \\[2ex] 0 & \text{if } \quad x = 9 \end{cases}$$

2. Lesson 2

2.1. Continuous functions defined on compact sets

2.1. Theorem *[the continuous image of a compact set is compact]*
Let $f \in \mathbb{R} \to \mathbb{R}$ be continuous function ($f \in C$) and suppose that D_f is compact. Then R_f is compact.

Proof. Let $y_n \in R_f$ $(n \in \mathbb{N})$ be a sequence in the range of f. Then $\exists\, x_n \in D_f$: $f(x_n) = y_n (n \in \mathbb{N})$. D_f is compact, so there exists a convergent subsequence (x_{ν_n}) whose limit – denoted by α – is in D_f. Using the Transference Theorem we obtain that

$$\lim_{n\to\infty} y_{\nu_n} = \lim_{n\to\infty} f(x_{\nu_n}) = f(\alpha) \in R_f \,.$$

So R_f is compact. □

Before stating the following theorem let us define the extreme values of a function:

2.2. Definition Let $f \in \mathbb{R} \to \mathbb{R}$. The minimum of f is the minimal element of its range (if exists) that is

$$\min f := \min R_f = \min \{f(x) \mid x \in D_f\} = \min_{x\in D_f} f(x) \,.$$

Respectively, the maximum of f is the maximal element of its range (if exists) that is

$$\max f := \max R_f = \max \{f(x) \mid x \in D_f\} = \max_{x\in D_f} f(x).$$

These numbers are called the absolute (or global) extreme values (absolute (or global) minimum, absolute (or global) maximum) of f.

2.3. Theorem *[Theorem of Weierstrass] Let $f \in \mathbb{R} \to \mathbb{R}$, $f \in C$, D_f compact. Then $\exists \min f$ and $\exists \max f$.*

Proof. By the previous theorem R_f is a compact set in $\mathbb{R}$, then – by the 1.17 Theorem – $\exists \min R_f$ and $\exists \max R_f$. □

In the following definition we give a stronger variation of continuity, whose essence is that the number δ in the definition of continuity is independent of the place.

2.4. Definition Let $f \in \mathbb{R} \to \mathbb{R}$. We say that f is uniformly continuous if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x,\, y \in D_f,\; |x - y| < \delta : \quad |f(x) - f(y)| < \varepsilon \,.$$

2.5. Remark. The definition of continuity of f means that $\forall\, y \in D_f :\ f \in C(y)$ that is

$$\forall\, y \in D_f \quad \forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x \in D_f,\ |x - y| < \delta : \quad |f(x) - f(y)| < \varepsilon\,.$$

Comparing these two definitions one can see that in the case of continuity δ depends on both y and ε that is $\delta = \delta(y, \varepsilon)$ but in the case of uniform continuity δ depends only on ε and is independent from the place y that is $\delta = \delta(\varepsilon)$.

Obviously every uniformly continuous function is continuous. The converse of this statement is false: there exists a continuous but not uniformly continuous function. An example for such function is:

$$f : (0, +\infty) \to \mathbb{R},\ x \mapsto \frac{1}{x}\,.$$

The following theorem – that is stated without proof – gives a sufficient condition for uniform continuity.

2.6. Theorem *[Theorem of Heine about the uniform continuity] Let $f \in \mathbb{R} \to \mathbb{R}$, $f \in C$, D_f compact. Then f is uniformly continuous.*

The concept of uniform continuity on a set is defined via restriction:

2.7. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $\emptyset \neq H \subseteq D_f$. We say that f is uniformly continuous on the set H if the restricted function $f_{|H}$ is uniformly continuous.

2.2. Continuous functions defined on intervals

Review: Let $a, b \in \mathbb{R}$, $a < b$. The intervals with endpoints a and b are the well-known sets:

$[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$: closed interval;

$[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\}$: interval closed from the left and open from the right;

$(a, b] := \{x \in \mathbb{R} \mid a < x \leq b\}$: interval open from the left and closed from the right;

$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$: open interval.

Moreover we can define the nonbounded intervals:

$[a, +\infty) := \{x \in \mathbb{R} \mid x \geq a\}$;

$(a, +\infty) := \{x \in \mathbb{R} \mid x > a\}$;

$(-\infty, b] := \{x \in \mathbb{R} \mid x \leq b\}$;

$(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}$;

$(-\infty, +\infty) := \mathbb{R}$.

The number "a" is called the left hand endpoint (or: starting point) of the interval and the number "b" is called the right hand endpoint (or: terminal point) of the interval.

It can be proved that a nonepty set $H \subseteq \mathbb{R}$ is interval if and only if $(\inf H, \sup H) \subseteq H$.

2.8. Theorem *[Intermediate Value Theorem, Theorem of Bolzano] Let $f : [a, b] \to \mathbb{R}$, $f \in C$. Suppose that $f(a) \neq f(b)$, for example $f(a) < f(b)$ (the discussion of the case $f(a) > f(b)$ is similar). Then*

$$\forall\, c \in (f(a), f(b)) \quad \exists\, \xi \in (a, b) : \quad f(\xi) = c\,.$$

Proof. Let

$$x_0 := a, \quad y_0 := b, \quad z := \frac{x_0 + y_0}{2}.$$

If $f(z) = c$, then $\xi := z$ and the proof is ready.

If $f(z) < c$ then $x_1 := z,\ y_1 := y_0$.

If $f(z) > c$ then $x_1 := x_0,\ y_1 := z$.

For the interval $[x_1,\, y_1]$ we have

$$[x_1,\, y_1] \subset [x_0,\, y_0], \quad y_1 - x_1 = \frac{y_0 - x_0}{2}, \quad f(x_1) < c < f(y_1).$$

Similarly we can define recursively the interval $[x_{n+1}, y_{n+1}]$ from $[x_n, y_n]$. So – if the process does not stop at some step – we obtain a sequence of intervals $([x_n,\, y_n])$ for which

$$x_0 \leq x_1 \leq x_2 \leq \ldots \leq y_2 \leq y_1 \leq y_0, \qquad y_n - x_n = \frac{y_0 - x_0}{2^n}$$

$$\text{and} \qquad f(x_n) < c < f(y_n) \quad (n \in \mathbb{N})\,.$$

The sequence (x_n) is monotonically increasing and bounded above so it is converges to a number α. Respectively the sequence (y_n) is monotonically decreasing and bounded below so it converges to the number β.

Using the connection between the limit and the ordering relations follows:

$$0 \leq \beta - \alpha \leq y_n - x_n = \frac{y_0 - x_0}{2^n} \to 0 \quad (n \to \infty).$$

From here follows that $\alpha = \beta =: \xi$. The continuity of f at ξ implies – using the Transference Theorem – that

$\lim f(x_n) = \lim f(y_n) = f(\xi)$. Using this fact let us make $n \to \infty$ in the following inequality

$$f(x_n) < c < f(y_n) \quad (n \in \mathbb{N})$$

which implies $f(\xi) \leq c \leq f(\xi)$ that is $f(\xi) = c$. □

2.9. Corollary. If $f(a) \cdot f(b) < 0$ then the equation $f(x) = 0$ has at least one root in the interval (a, b) and this root can be approximated by the sequences (x_n) and (y_n) defined above. The speed of the convergence is $(\frac{1}{2})^n$ (see: Numerical Analysis).

2.10. Theorem *[The continuous image of an interval is interval] Let $f \in \mathbb{R} \to \mathbb{R}$, $f \in C$and suppose that D_f is an interval. Then R_f is also an interval.*

Proof. It is enough to prove that $(\inf R_f, \sup R_f) \subseteq R_f$.

Let $\inf R_f < y < \sup R_f$. By the definitions of sup and inf it follows

$$\exists\, y_1 \in R_f : \quad f(x_1) = y_1 < y \quad \text{and} \quad \exists\, y_2 \in R_f : \quad f(x_2) = y_2 > y,$$

where $x_1,\ x_2$ are suitable elements in D_f. $f(x_1) < y < f(x_2)$ therefore using Bolzano's theorem we obtain:

$$\exists\, \xi \in (x_1, x_2) \subseteq I : \quad f(\xi) = y\,.$$

Consequently $y \in R_f$. □

2.3. Monotone and continuous functions defined on intervals

About the definition of monotonity and about the limit of monotone functions see: Analysis-1.

2.11. Theorem *Let $I \subseteq \mathbb{R}$ be an interval with starting point a and with terminal point b. Let $f : I \to \mathbb{R}$ be continuous. Suppose that f is monotonically increasing ($\nearrow$). Then the starting point of R_f is $\lim\limits_{a+} f$ and the terminal point of R_f is $\lim\limits_{b-} f$.*

Moreover if f is strictly monotonically increasing ($\uparrow$) then its inverse function f^{-1} is also strictly monotonically increasing ($\uparrow$) and continuous.

2.12. Remarks.

1. If f is continuous and strictly increasing ($\uparrow$) then the following cases may occur:

$I = D_f$	R_f
$[a, b]$	$[f(a), f(b)]$
$(a, b]$	$(\lim\limits_{a+} f, f(b)]$
$[a, b)$	$[f(a), \lim\limits_{b-} f)$
(a, b)	$(\lim\limits_{a+} f, \lim\limits_{b-} f)$

2. A similar theorem can be stated if f is monotonically decreasing ($\searrow$) or strictly monotonically decreasing ($\downarrow$).

The following theorem can be proved also by means of Bolzano's theorem:

2.13. Theorem *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be continuous. Suppose that f is one-to-one (injective). Then f is strictly monotone ($\uparrow$ or $\downarrow$) function.*

2.4. The real exponential and logarithm functions

Review: In Analysis-1 we have learned about the exponential function of real or complex variable. In this section let us look at the real variable case:

$$\exp x := 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (x \in \mathbb{R}) .$$

We have seen in Analysis-1 that

$$\exp 0 = 1, \quad \exp(x+y) = \exp x \cdot \exp y, \quad \exp(-x) = \frac{1}{\exp x} .$$

From the continuity of analytical functions it follows immediately that exp is a continuous function.

From the power series expansion of exp the below properties follow immediately:

1. $\forall\, x > 0 : \quad \exp x > 1.$
2. $\forall\, x < 0 : \quad 0 < \exp x < 1.$
3. $\lim\limits_{x \to +\infty} \exp x = +\infty.$
4. $\lim\limits_{x \to -\infty} \exp x = 0.$

2.14. Theorem $\exp : \mathbb{R} \to \mathbb{R}$ *is a strictly monotonically increasing* ($\uparrow$) *function.*

Proof. Let $x, y \in \mathbb{R},\ x < y$. Then $y - x > 0$ so $\exp(y-x) > 1$. Therefore

$$\exp y = \exp\left((y-x) + x\right) = \exp(y-x) \cdot \exp(x) > 1 \cdot \exp x = \exp x.$$

$\square$

Using the 2.11 Theorem we have that $R_{exp} = (0, +\infty)$.

2.15. Definition The inverse function of the real exponential function is called natural logarithm function and is denoted by ln. So

$$\ln := \exp^{-1} .$$

From this definition one can simply deduce the following basic properties of the natural logarithmic function:

$$D_{\ln} = R_{\exp} = (0, +\infty), \quad R_{\ln} = D_{\exp} = \mathbb{R}, \quad \ln 1 = 0, \quad \ln(xy) = \ln x + \ln y ,$$

ln is strictly monotonically increasing, $\lim\limits_{x \to +\infty} \ln x = +\infty$, $\lim\limits_{x \to 0-0} \ln x = -\infty$.

In the next part the connection between the real exponential function and the powers of e is discussed.

2.16. Theorem $\forall\, r \in \mathbb{Q}: \quad \exp r = e^r$

where e denotes the Euler-number $e = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$.

Proof. In Analysis-1 we have seen that $\sum\limits_{n=0}^{\infty} \dfrac{1}{n!} = e$. We can write for any $p, q \in \mathbb{N}$:

$$\left(\exp\left(\frac{p}{q}\right)\right)^q = \exp\left(\sum_{i=1}^{q} \frac{p}{q}\right) = \exp\left(q \cdot \frac{p}{q}\right) =$$

$$= \exp p = \exp\left(\sum_{i=1}^{p} 1\right) = (\exp 1)^p = e^p\,.$$

So $\exp\left(\dfrac{p}{q}\right) = e^{p/q}$, and the theorem is proved for positive rational numbers.

In the case $r \in \mathbb{Q}$, $r < 0$ we write (using $-r > 0$):

$$\exp r = \exp(-(-r)) = \frac{1}{\exp(-r)} = \frac{1}{e^{-r}} = e^r\,.$$

The case $r = 0$ is trivial. □

2.17. Definition Let $x \in \mathbb{R}$. Then the power e^x is defined as $e^x := \exp x$.

2.18. Remark. $\exp x$ is the unique continuous extension of e^x from $\mathbb{Q}$ to $\mathbb{R}$. This fact is based on the density of $\mathbb{Q}$ in $\mathbb{R}$ (see: Analysis-1).

2.19. Definition Let $a > 0$. Then the exponential function with the base a is defined as follows:

$$\exp_a x := \exp(x \cdot \ln a) \qquad (x \in \mathbb{R})\,.$$

Remark that $\exp_e x = \exp x$. It can be proved – similarly to the exp function – that

$$\forall\, r \in \mathbb{Q}: \quad \exp_a r = a^r\,.$$

This fact motivates the definition $a^x := \exp_a x$ for every $x \in \mathbb{R}$ (unique continuous extension from $\mathbb{Q}$ to $\mathbb{R}$).

One can see easily that the function $\mathbb{R} \ni x \mapsto \exp_a x$ is invertable if and only if $a \neq 1$.

2.20. Definition Let $a > 0$, $a \neq 1$. The inverse function of the exponential function with the base a is called the logarithm function with base a and is denoted by $\log_a$. So $\log_a := \exp_a^{-1}$.

Remark that $\log_e x = \ln x$.

Finally let us observe that the power functions with fixed exponent $\mu \in \mathbb{R}$ can be written in the form:

$$x^\mu = \exp_x \mu = \exp(\mu \cdot \ln x) \qquad (x \in \mathbb{R},\ x > 0)\,.$$

All the usual rules and identities in connection with the powers and logarithms that we have learned in the secondary school can be proved.

2.5. Homeworks

Prove that the given equations have roots in the given intervals. Is this root unique?

1. $x^3 - 3x + 1 = 0$ in the interval $(0,\ 1)$. Compute the first 3 terms of the sequence that approximates the root. Estimate the error of approximation with this 3-rd term.
2. $\ln x = e^{-x}$ in the interval $(1,\ e)$.
3. $\cos x = x$ in the interval $(0,\ 1)$.

3. Lesson 3

3.1. Differentiation of functions

3.1. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in intD_f$. f is differentiable at "a" $\overset{\text{df}}{\Leftrightarrow}$

$$\exists \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \in \mathbb{R}\,.$$

In this case $f'(a) := \lim\limits_{a} \dfrac{f(x) - f(a)}{x - a}$. This number is called the derivative of f at the point "a".

Let us denote the set of functions that are differentiable at "a" by $D(a)$.

3.2. Remarks.

1. Other notations for $f'(a)$ are: $\left(\dfrac{df}{dx}\right)_{|x=a}$, $(f(x))'_{|x=a}$.
2. The geometrical meaning of the derivative is: the slope of the tangent line to the graph of f at the point $(a,\, f(a))$.
3. The physical meaning of the derivative is: the instantaneous velocity of a process (e.g. a motion).

Using the substitution $h = x - a$ we obtain an equivalent form:

$$f'(a) := \lim_{a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}\,.$$

The last expression is useful because the letter x became „free" so we can write the definition so:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}\,.$$

3.3. Definition Let $f \in \mathbb{R} \to \mathbb{R}$ and suppose that the set

$$D_{f'} := \{x \in intD_f \mid f \in D(x)\}$$

is nonempty. Then the function

$$f' : D_{f'} \to \mathbb{R}, \quad x \mapsto f'(x)$$

is called the derivative function (or simply: the derivative) of f.

If $D_{f'} = intD_f \neq \emptyset$ then we say that the function f is differentiable and denote this fact by $f \in D$.

3.4. Theorem $f \in D(a) \Rightarrow f \in C(a)$.

Proof. The difference $f(x) - f(a)$ tends to 0 since:

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a} \cdot (x - a) \to f'(a) \cdot 0 = 0 \quad (x \to a)$$

□

Remark that the opposite statement is not true. For example let us take the continuous function $f(x) = |x|$ at 0:

$$\lim_{x \to 0+} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0+} \frac{|x|}{x} = \lim_{x \to 0+} \frac{x}{x} = \lim_{x \to 0+} 1 = 1\,,$$

$$\lim_{x \to 0-} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0-} \frac{|x|}{x} = \lim_{x \to 0+} \frac{-x}{x} = \lim_{x \to 0+} (-1) = -1\,.$$

So $\lim\limits_{x \to 0} \frac{|x|}{x}$ does not exist, i.e. $f \notin D(0)$.

3.2. Some basic derivatives

In this section we compute some important basic derivatives using the definition.

1. $f(x) := c$ where $c \in \mathbb{R}$ is fixed (the constant function). Then $\forall\, x \in \mathbb{R}$:

$$f'(x) = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0\,.$$

2. $f(x) = ax + b$ where $a, b \in \mathbb{R}$ are fixed (the linear function). Then $\forall\, x \in \mathbb{R}$:

$$f'(x) = \lim_{h \to 0} \frac{a \cdot (x + h) + b - a \cdot x - b}{h} = \lim_{h \to 0} \frac{ax + ah + b - ax - b}{h} = \lim_{h \to 0} a = a\,,$$

especially $(x)' = 1$.

3. $f(x) = x^n$ where $n \in \mathbb{N}$ is fixed. Then – using the binomial theorem – $\forall\, x \in \mathbb{R}$:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x + h)^n - x^n}{h} = \\ &= \lim_{h \to 0} \frac{\binom{n}{0} \cdot x^n + \binom{n}{1} \cdot x^{n-1} \cdot h + \binom{n}{2} \cdot x^{n-2} \cdot h^2 + \ldots + \binom{n}{n} \cdot h^n - x^n}{h} = \\ &= \lim_{h \to 0} \left(\binom{n}{1} \cdot x^{n-1} + \binom{n}{2} \cdot x^{n-2} \cdot h + \ldots + \binom{n}{n} \cdot h^{n-1} \right) = \\ &= \binom{n}{1} \cdot x^{n-1} = n \cdot x^{n-1}\,. \end{aligned}$$

4. $f(x) = e^x$ (the exponential function). Then $\forall\, x \in \mathbb{R}$:

$$f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h} = \lim_{h\to 0} e^x \cdot \frac{e^h - 1}{h} =$$
$$= e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x\,.$$

   Here we have used the familiar limit $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$ (see: Practice).

5. $f(x) = \sin x$ (the sinus function). Then $\forall\, x \in \mathbb{R}$:

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h} =$$
$$= \lim_{h\to 0} \left(\cos x \cdot \frac{\sin h}{h} - \sin x \cdot \frac{1 - \cos h}{h^2} \cdot h \right) = \cos x.$$

   Here we have used the familiar limits $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$ (see: Practice).

6. $(\cos x)' = -\sin x$ $(x \in \mathbb{R})$ can be proved similarly.

3.3. Differentiation Rules

1. Sum

3.5. Theorem *Let $f, g \in D(x)$. Then $f + g \in D(x)$ and*

$$(f+g)'(x) = f'(x) + g'(x)\,.$$

Proof. It can be proved that $x \in intD_{f+g}$. To see the derivative of $f + g$ let us compute as follows:

$$(f+g)'(x) = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} =$$
$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} =$$
$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x)\,.$$

□

2. Product

3.6. Theorem *Let $f, g \in D(x)$. Then $fg \in D(x)$ and*

$$(fg)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)\,.$$

Proof. It can be proved that $x \in intD_{fg}$. To see the derivative of fg let us compute as follows:

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h} = \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h} = \\
&= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x+h) + f(x)\cdot g(x+h) - f(x)\cdot g(x)}{h} = \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \\
&= f'(x)g(x) + f(x)g'(x)\,.
\end{aligned}$$

□

The scalar multiple special case: Apply the Product Rule with the constant function $g(x) = c$ to obtain: $(c \cdot f(x))' = c \cdot f'(x)$.

3. Quotient

3.7. Theorem *Let $f, g \in D(x)$, $g(x) \neq 0$. Then $\frac{f}{g} \in D(x)$ and*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}\,.$$

Proof. It can be proved that $x \in intD_{f/g}$. To see the derivative of $\frac{f}{g}$ let us compute as follows:

$$\begin{aligned}
\left(\frac{f}{g}\right)'(x) &= \lim_{h\to 0} \frac{(\frac{f}{g})(x+h) - (\frac{f}{g})(x)}{h} = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \\
&= \lim_{h\to 0} \frac{f(x+h)\cdot g(x) - f(x)\cdot g(x+h))}{h\cdot g(x)\cdot g(x+h)} = \\
&= \lim_{h\to 0} \frac{f(x+h)\cdot g(x) - f(x)\cdot g(x) + f(x)\cdot g(x) - f(x)\cdot g(x+h)}{h\cdot g(x)\cdot g(x+h)} = \\
&= \left[\lim_{h\to 0} \frac{1}{g(x)\cdot g(x+h)}\right] \cdot \left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x) - f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right] = \\
&= \frac{1}{g(x)\cdot g(x)} \cdot \left[f'(x)\cdot g(x) - f(x)\cdot g'(x)\right] = \\
&= \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}\,.
\end{aligned}$$

□

Special case: the reciprocical: Apply the Quotient Rule with the constant function $f(x) = 1$. We obtain: $\left(\dfrac{1}{g(x)}\right)' = -\dfrac{g'(x)}{(g(x))^2}$.

4. Composition (Chain Rule) without proof

3.8. Theorem *Let $g \in \mathbb{R} \to \mathbb{R}$, $g \in D(x)$, $f \in \mathbb{R} \to \mathbb{R}$, $f \in D(g(x))$. Then $f \circ g \in D(x)$ and*

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)\,.$$

5. Inverse Rule without proof

3.9. Theorem *Let $I \subseteq R$ be an open interval, $f : I \to \mathbb{R}$, $f \in D$, be an (strictly) increasing function. Furthermore suppose that $f'(x) \neq 0$ $(x \in I)$. Then $f^{-1} \in D(J)$ where $J = R_f$ (we know that R_f is an open interval) and*

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \quad (y \in J)\,.$$

3.4. Some other basic derivatives

Using the Differentiation Rules we can deduce the derivatives of some basic functions.

1. $f(x) := \operatorname{tg} x$ (the tangent function). Then – using the Quotient Rule – $\forall\, x \in D_{\operatorname{tg}} = \mathbb{R} \setminus \{\frac{\pi}{2} + k \cdot \pi \mid k \in \mathbb{Z}\}$:

$$\operatorname{tg}' x = \left(\frac{\sin x}{\cos x}\right)' = \frac{\sin' x \cdot \cos x - \sin x \cdot \cos' x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = 1 + \operatorname{tg}^2 x.$$

2. $g(x) := \ln x$ $(x > 0)$ (the natural logarithm function).

   Let $f(x) = e^x = \exp x$ $(x \in \mathbb{R})$. Then $f^{-1}(y) = \ln y$ $(y \in \mathbb{R}^+)$. All the assumptions of the Inverse Rule are satisfied, so:

$$\ln'(y) = (f^{-1})'(y) = \frac{1}{\exp'(\ln(y))} = \frac{1}{\exp(\ln(y))} = \frac{1}{y} \quad (y \in J = \mathbb{R}^+)\,.$$

   If "y" is exchanged for "x": $(\ln x)' = \dfrac{1}{x}$ $(x \in \mathbb{R}^+)$.

3.5. Homeworks

1. Compute by definition the derivative of $f(x) = \dfrac{1}{2x-1}$ at the point $x_0 = 3$.

2. Determine the derivatives of

$$a)\quad f(x) = \frac{4x+3}{\sqrt{x^2+5}} \qquad b)\quad f(x) = \ln \operatorname{tg} \sin \cos x$$

$$c)\quad f(x) = (x^2+2)\sin\sqrt{x+3} \qquad d)\quad f(x) = \frac{\operatorname{tg} x}{1+\operatorname{tg}^2 x}$$

3. Determine the equation of the tangent line to the given curve at its given point (only the first coordinate x_0 of the point is given):

$$y = \frac{1}{\ln^2(x - \frac{1}{x})}, \qquad x_0 = 2\,.$$

4. Lesson 4

4.1. Local extrema of functions

In connection with the Weierstrass-theorem (see: 2.3 Theorem) we have defined the (global or absolute) extreme values of a function. Now we will discuss the so called local extrema.

4.1. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in D_f$. We say that f has at "a"

1. local minimum $\overset{\text{df}}{\Leftrightarrow} \exists\, r > 0\ \forall\, x \in B(a,r) \cap D_f :\ f(x) \geq f(a)$;
2. strict local minimum $\overset{\text{df}}{\Leftrightarrow} \exists\, r > 0\ \forall\, x \in B(a,r) \cap D_f \setminus \{a\} :\ f(x) > f(a)$;
3. local maximum $\overset{\text{df}}{\Leftrightarrow} \exists\, r > 0\ \forall\, x \in B(a,r) \cap D_f :\ f(x) \leq f(a)$;
4. strict local maximum $\overset{\text{df}}{\Leftrightarrow} \exists\, r > 0\ \forall\, x \in B(a,r) \cap D_f \setminus \{a\} :\ f(x) < f(a)$;

Here "a" is the point of the local extremum and $f(a)$ is the local extreme value.

4.2. Theorem *[First Derivative Test for local extremum]*
Let $f \in \mathbb{R} \to \mathbb{R}$, $f \in D(a)$ and suppose that f has a local extremum at a. Then $f'(a) = 0$.

Proof. Suppose indirectly that $f'(a) \neq 0$. Then either $f'(a) > 0$ or $f'(a) < 0$. Take for example the case $f'(a) > 0$ (the other case can be discussed similarly). By the definition of the derivative:

$$0 < f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}\,.$$

It follows from the definition of the limit that

$$\exists\, \delta > 0\ \forall\, x \in (a - \delta, a + \delta) \setminus \{a\} : \quad \frac{f(x) - f(a)}{x - a} > \frac{f'(a)}{2} > 0\,.$$

Since $(a - \delta, a + \delta) \setminus \{a\} = (a - \delta, a) \cup (a, a + \delta)$ let us discuss two cases: $x < a$, $x > a$.

First let $x \in (a - \delta, a)$. In this case $x - a < 0$, so from the sign of fraction follows that $f(x) - f(a) < 0$ that is $f(x) < f(a)$.

Similarly, if $x \in (a, a + \delta)$ then $x - a > 0$, so – by the sign of the fraction – $f(x) - f(a) > 0$ that is $f(x) > f(a)$.

Since any neigbourhood of "a" contain both types of these points the function f has no extreme value at "a". □

4.3. Remarks. 1. The reverse of the theorem is not true, see for example the function $f(x) = x^3$ ($x \in \mathbb{R}$) at $a = 0$.

2. If $f \in \mathbb{R} \to \mathbb{R}$, $f \in D$ then the points of local extrema are contained in the set of the roots of the equation $f'(x) = 0$. The roots of $f'(x) = 0$ are called critical points or stationary points.

4.2. Mean Value Theorems

4.4. Theorem *[Rolle]*
Let $f : [a, b] \to \mathbb{R}$, $f \in C$, $f \in D$. Suppose that $f(a) = f(b)$.
Then $\exists\, \xi \in (a, b) :\ f'(\xi) = 0$.

Proof. By the Weierstrass-theorem $\exists \min f$ and $\exists \max f$.

If $\min f = \max f$ then f is constant so every $\xi \in (a, b)$ is a good choice.

If $\min f < \max f$ then – using $f(a) = f(b)$ one of them is taken in the inside of $[a,\ b]$ that is at a certain $\xi \in (a,\ b)$. So the First Derivative Test can be applied: $f'(\xi) = 0$. □

4.5. Theorem *[Cauchy] Let $f, g : [a, b] \to \mathbb{R}$, $f, g \in C$, $f, g \in D$. Suppose that $g'(x) \neq 0$ ($x \in (a, b)$). Then*

$$\exists\, \xi \in (a, b) : \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}\,.$$

Proof. Let $F(x) = f(x) + \lambda g(x)$. We want to apply the Rolle-theorem for F therefore we choose the parameter λ so hat $F(a) = F(b)$ holds.

$$F(a) = f(a) + \lambda g(a) = f(b) + \lambda g(b) = F(b)$$

$$\lambda = \frac{f(b) - f(a)}{g(a) - g(b)} = -\frac{f(b) - f(a)}{g(b) - g(a)}\,.$$

By the Rolle-theorem $\exists\, \xi \in (a, b) :\ F'(\xi) = 0$. So

$$f'(\xi) - \frac{f(b) - f(a)}{g(b) - g(a)} \cdot g'(\xi) = 0.$$

The statement of the theorem can be obtained by rearranging this equation. □

4.6. Theorem *[Lagrange] Let $f : [a, b] \to \mathbb{R}$, $f \in C$, $f \in D$. Then*

$$\exists\, \xi \in (a, b) : \quad \frac{f(b) - f(a)}{b - a} = f'(\xi).$$

Proof. Let us apply the Cauchy-theorem with $g(x) = x$ ($x \in [a, b]$):

$$\exists\, \xi \in (a, b) : \quad \frac{f(b) - f(a)}{b - a} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)} = \frac{f'(\xi)}{1} = f'(\xi)\,.$$

□

4.3. Discussion of Monotonity

The monotonity of functions defined on intervals can be effectively discussed using derivatives.

4.7. Theorem *[First Derivative Test for monotonity] Let $I \subseteq \mathbb{R}$ be an interval (of any type), $f : I \to \mathbb{R}$, $f \in C$, $f \in D$. Then*

1. *If $\forall\, x \in intI :\ f'(x) > 0$ then f is strictly increasing (on I).*
2. *If $\forall\, x \in intI :\ f'(x) < 0$ then f is strictly decreasing (on I).*
3. *If $\forall\, x \in intI :\ f'(x) = 0$ then f is constant (on I).*

Proof.

Let $x_1, x_2 \in I,\ x_1 < x_2$ and let us apply the Lagrange-theorem on the closed interval $[x_1, x_2]$:

$$\exists\, \xi \in (x_1, x_2) : \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi)\,.$$

After rearrangement: $f(x_2) - f(x_1) = f'(\xi) \cdot (x_2 - x_1)$.

1. Since $\xi \in intI$ we have $f'(\xi) > 0$. Then $x_2 - x_1 > 0$ implies $f(x_2) - f(x_1) > 0$ that is $f(x_1) < f(x_2)$.
2. Since $\xi \in intI$ we have $f'(\xi) < 0$. Then $x_2 - x_1 > 0$ implies $f(x_2) - f(x_1) < 0$ that is $f(x_1) > f(x_2)$.
3. Since $\xi \in intI$ we have $f'(\xi) = 0$. Therefore $f(x_2) - f(x_1) = 0$ that is $f(x_1) = f(x_2)$.

□

4.8. Remark. The practical application of the theorem: Let $f \in \mathbb{R} \to \mathbb{R}$, D_f be interval. Suppose that the equation $f'(x) = 0$ has finite many roots: $x_1 < x_2 < \ldots < x_k$. If f' is continuous then the sign of f' is constant on the interval (x_{j-1}, x_j). Consequently the function is strictly increasing or strictly decreasing over $[x_{j-1}, x_j]$.

4.4. Inverse trigonometric functions

4.9. Theorem *There exists a unique number $\alpha \in (0, 2)$ such that* $\cos\alpha = 0$.

Proof. For the existence it is enough to prove that $\cos 0 > 0$ and $\cos 2 < 0$. From here – using the continuity of cos and the Bolzano-theorem – the existence follows. Indeed, $\cos 0 = 1 > 0$. On the other hand:

$$\cos 2 = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \frac{2^8}{8!} - \ldots = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} \cdot \underbrace{(1 - \frac{2^2}{7 \cdot 8})}_{+} - \ldots -$$

$$- \frac{2^{2n+2}}{(2n+2)!} \cdot \underbrace{(1 - \frac{2^2}{(2n+3)(2n+4)})}_{+} - \ldots < 1 - \frac{2^2}{2!} + \frac{2^4}{4!} = -\frac{1}{3} < 0\,.$$

For the uniqueness it is enough to prove that the cos function is strictly decreasing over the interval $[0,\, 2]$. Since the derivative of cos is $-\sin$, it is enough to show that $\sin x > 0$ for any $0 < x < 2$. Really, if $0 < x < 2$ then

$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots = \\ &= x \cdot (1 - \frac{x^2}{2 \cdot 3}) + \frac{x^5}{5!} \cdot (1 - \frac{x^2}{6 \cdot 7}) + \frac{x^9}{9!} \cdot (1 - \frac{x^2}{10 \cdot 11}) + \ldots > \\ &> x \cdot \underbrace{(1 - \frac{2^2}{2 \cdot 3})}_{+} + \frac{x^5}{5!} \cdot \underbrace{(1 - \frac{2^2}{6 \cdot 7})}_{+} + \frac{x^9}{9!} \cdot \underbrace{(1 - \frac{2^2}{10 \cdot 11})}_{+} + \ldots > 0\,.\end{aligned}$$

□

4.10. Definition Let $\pi := 2\alpha$ where α is the number in the previous theorem.

Since $\alpha = \frac{\pi}{2}$ is the unique zero of cos in $[0,\, 2]$ and $\cos 0 = 1 > 0$ we can have that $\cos x > 0$ if $0 \le x < \frac{\pi}{2}$. The cos function is even, so $\cos x > 0$ for every $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Since $(\sin x)' = \cos x$ there follows that the sin function is strictly increasing over the interval $\left[-\frac{\pi}{2},\, \frac{\pi}{2}\right]$. Let us compute $\sin \frac{\pi}{2}$:

$$\sin^2 x + \cos^2 x = 1 \quad \text{and} \quad \sin \frac{\pi}{2} > 0 \quad \text{imply} \quad \sin \frac{\pi}{2} = \sqrt{1 - \cos^2 \frac{\pi}{2}} = \sqrt{1 - 0} = 1\,.$$

The sin function is odd, so $\sin(-\frac{\pi}{2}) = -\sin \frac{\pi}{2} = -1$. Therefore the range of the restricted $\sin\big|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$ function is $[-1,\, 1]$.

By the previous facts we can state that the restricted $\sin\big|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$ function has inverse which is called arcsin.

4.11. Definition $\arcsin := \sin^{-1}\Big|_{[-\frac{\pi}{2},\frac{\pi}{2}]} : [-1,1] \to \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

4.12. Remark.

$$\arcsin y = x \Leftrightarrow x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \quad \text{and} \quad \sin x = y\,.$$

The derivative of arcsin:

Let $f(x) = \sin x$ $(x \in I := \left(-\frac{\pi}{2},\frac{\pi}{2}\right))$. In this case $f^{-1}(y) = \arcsin y$ $(y \in (-1\,1))$. We can apply the theorem about the derivative of inverse function:

$$\arcsin' y = (f^{-1})'(y) = \frac{1}{\sin'(\arcsin y)} = \frac{1}{\cos(\arcsin y)}.$$

Since $\arcsin y \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ it is obvious that $\cos(\arcsin y) > 0$, so we can continue as follows:

$$\arcsin' y = \frac{1}{\sqrt{1-(\sin(\arcsin y))^2}} = \frac{1}{\sqrt{1-y^2}}.$$

So $\arcsin' y = \dfrac{1}{\sqrt{1-y^2}}$ $\quad (y \in (-1,1))$. After replacing "y" by "x":

$$\arcsin' x = \frac{1}{\sqrt{1-x^2}} \quad (x \in (-1,1))\,.$$

Using similar consideration we can define the inverses of cos, tg and ctg. Here is the collection of them and their derivatives:

4.13. Definition $\arccos := \cos^{-1}\Big|_{[0,\pi]} : [-1,1] \to [0,\pi]$

$\operatorname{arc\,tg} := \operatorname{tg}^{-1}\Big|_{(-\frac{\pi}{2},\frac{\pi}{2})} : \mathbb{R} \to \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\operatorname{arc\,ctg} := \operatorname{ctg}^{-1}\Big|_{(0,\pi)} : \mathbb{R} \to (0,\pi)$

Their derivatives can be computed like the one of the arcsin. The results:

$$\arccos' x = -\frac{1}{\sqrt{1-x^2}} \quad (x \in (-1,1))\,;$$

$$\operatorname{arc\,tg}' x = \frac{1}{1+x^2} \quad (x \in \mathbb{R})\,;$$

$$\operatorname{arc\,ctg}' x = -\frac{1}{1+x^2} \quad (x \in \mathbb{R})\,.$$

4.5. Homeworks

1. On what intervals is f increasing and decreasing? At what points has it local extreme value?

$$a)\quad f(x)=x^3-3x^2 \qquad b)\quad f(x)=\frac{x^2}{(x-1)^2}$$

$$c)\quad f(x)=\frac{x}{x^2-6x-16} \qquad d)\quad f(x)=x\cdot e^{-x}$$

2. A right triangle whose hypotenuse is $\sqrt{3}$ long is revolved about one of its legs to generate a right circular cone. Find the radius and height of the cone of greatest volume.

3. Find the absolute extreme values (and their places) of

$$f(x)=\frac{x}{x^2+x+1} \qquad (-2\le x\le 0)\,.$$

4. Prove that $\operatorname{arc\,tg}' x=\dfrac{1}{1+x^2} \quad (x\in\mathbb{R})$.

5. Lesson 5

5.1. The L'Hospital Rule

An important application of the differential calculus is the computation of indeterminate form limits via the L'Hospital Rule:

5.1. Theorem *[L'Hospital Rule] Let* $-\infty \leq a < b \leq +\infty$, $f, g : (a, b) \to \mathbb{R}$, $f, g \in D$, $g'(x) \neq 0$ $(x \in (a, b))$. *Suppose that*

either $\lim\limits_{a+0} f = \lim\limits_{a+0} g = 0$ *or* $\lim\limits_{a+0} f = \lim\limits_{a+0} g = +\infty$ *and that* $\exists \lim\limits_{a+0} \dfrac{f'}{g'}$.

Then

$$\lim_{a+0} \frac{f}{g} = \lim_{a+0} \frac{f'}{g'}.$$

Proof. We will prove only that part of the case $\dfrac{0}{0}$ when $a > -\infty$.

Let $A := \lim\limits_{a+0} \dfrac{f'}{g'}$ and $\varepsilon > 0$. This implies – by the definition of limit:

$$\exists\, \delta > 0 : \quad a + \delta < b,\ \forall\, x \in (a, a + \delta) : \frac{f'(x)}{g'(x)} \in B(A, \varepsilon)\,.$$

Let $f(a) := g(a) := 0$ and take a number $x \in (a, a + \delta)$. Let us apply the Cauchy Mean Value Theorem on the interval $[a, x]$:

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi)}{g'(\xi)} \in B(A, \varepsilon)\,.$$

It means – by the definition of limit – that the statement of the theorem is true.

The other cases of the theorem can be proved similarly or can be reduced to the proved cases. □

Similar theorem can be proved for left-hand limits. From here it follows that a similar theorem is valid for limits. The indeterminate forms that are not $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ can be reduced via algebraic transforms to the indeterminate quotient case.

5.2. Taylor-polynomials

First we discuss the higher order derivatives. The second order derivative is defined as the derivative of the derivative function.

5.2. Definition Let $f \in \mathbb{R} \to \mathbb{R},\ a \in intD_f$. We say that f is 2 times differentiable at "a" (its notation is: $f \in D^2(a)$) if

$$\exists\, r > 0\ \forall\, x \in B(a,r):\quad f \in D(x) \quad \text{and} \quad f' \in D(a).$$

In this case the number $f''(a) := (f')'(a)$ is called the second derivative of f at the point "a".

Similarly can be defined the 3., 4., ... derivatives with recursion. Their notations are:

$$f'''(a),\ f''''(a),\ \ldots \quad \text{or} \quad f^{(3)}(a),\ f^{(4)}(a),\ \ldots$$

Generally if f is k times differentable at "a" then we denote this fact by $f \in D^k(a)$ and the k-th order derivative by $f^{(k)}(a)$.

5.3. Definition Let $f \in \mathbb{R} \to \mathbb{R}$ and suppose that the set

$$D_{f^{(k)}} := \left\{ x \in intD_f \ \mid\ f \in D^k(x) \right\}$$

is nonempty. Then the function

$$f^{(k)} : D_{f^{(k)}} \to \mathbb{R}, \quad x \mapsto f^{(k)}(x)$$

is called the k-th order derivative function (or simply: the k-th derivative) of f.

If $D_{f^{(k)}} = intD_f$ then we say that the function f is k times differentiable and denote this fact by $f \in D^k$.

5.4. Definition (Taylor polynomial) Let $f \in \mathbb{R} \to \mathbb{R},\ f \in D^n(a)$. The polynomial

$$T_n(x) := f(a) + \frac{f'(a)}{1!}\cdot(x-a) + \frac{f''(a)}{2!}\cdot(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n =$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\cdot(x-a)^k \qquad (x \in \mathbb{R})$$

is called the n-th Taylor-polynomial of f relative to the center a.

5.5. Remarks.

1. It is obvious that the degree of T_n is at most n that is $T_n \in \mathcal{P}_n$.
2. Obviously $T_n(a) = f(a)$.
3. $T_n'(x) = \sum\limits_{k=1}^{n} \dfrac{f^{(k)}(a)}{k!}\cdot k\cdot(x-a)^{k-1}$. Hence we have $T_n'(a) = f'(a)$.
4. Similarly – using mathematical induction – one can prove that $T_n^{(j)}(a) = f^{(j)}(a)$ $(j = 0, \ldots, n)$.

5.6. Theorem *[Taylor's formula]*

Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$, $f \in D^{n+1}$, $a \in I$. Then for every $x \in I \setminus \{a\}$ there exists a number ξ between a and x such that:

$$f(x) - T_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot (x-a)^{n+1}.$$

The right-hand side of this equation is called the Lagrangian remainder term.

Proof.

Let us introduce the following auxiliary functions:

$$F(z) := f(z) - T_n(z) \quad \text{and} \quad G(z) := (z-a)^{n+1} \quad (z \in I)\,.$$

One can easily compute that

$$F^{(j)}(a) = f^{(j)}(a) - T_n^{(j)}(a) = 0, \quad G^{(j)}(a) = 0 \qquad (j = 0, \dots, n)\,,$$

furthermore

$$F^{(n+1)}(x) = f^{(n+1)}(x), \quad G^{(n+1)}(x) = (n+1)! \qquad (x \in I).$$

Suppose that $x \in I$, $x > a$ (right-hand case), and let us apply the Cauchy Mean Value Theorem consecutively (first for F and G on the interval $[a, x]$, then for F' and G' on the interval $[a, \xi_1]$, etc.). Thus we obtain that there exist numbers $a < \xi_{n+1} < \xi_n < \ldots < \xi_2 < \xi_1 < x$ such that

$$\frac{F(x)}{G(x)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F'(\xi_1)}{G'(\xi_1)} = \frac{F'(\xi_1) - F'(a)}{G'(\xi_1) - G'(a)} = \frac{F''(\xi_2)}{G''(\xi_2)} =$$
$$= \ldots = \frac{F^{(n)}(\xi_n)}{G^{(n)}(\xi_n)} = \frac{F^{(n)}(\xi_n) - F^{(n)}(a)}{G^{(n)}(\xi_n) - G^{(n)}(a)} = \frac{F^{(n+1)}(\xi_{n+1})}{G^{(n+1)}(\xi_{n+1})} = \frac{f^{(n+1)}(\xi_{n+1})}{(n+1)!}\,.$$

Let $\xi := \xi_{n+1}$. We have obtained that

$$\exists\, \xi \in (a, x) : \quad \frac{F(x)}{G(x)} = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,.$$

By the definitions of F and G:

$$\frac{f(x) - T_n(x)}{(x-a)^{n+1}} = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,.$$

From here we can finish the proof by a simple rearrangement.

The left-hand case $x < a$ can be proved similarly. □

5.7. Remark. In the case $n = 0$, $x = b > a$ (where $[a, b] \subset I$) the Taylor's Formula coincides with the Lagrange Mean Value Theorem.

5.3. Concavity

5.8. Definition Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$.

(a) f is called to be concave up (or: convex) if

$$\forall\, x, y \in I, x < y \;\forall\, 0 < \lambda < 1 : \quad f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y)\,;$$

(b) f is called to be concave down (or: concave) if

$$\forall\, x, y \in I, x < y \;\;\forall\, 0 < \lambda < 1 : \quad f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda) f(y)\,.$$

The following theorem can be proved:

5.9. Theorem *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, $f \in C$, $f \in D$. Then*
(a) f is concave up if and only if f' is strictly increasing;
(b) f is concave down if and only if f' is strictly decreasing.

Remark that the statement of the theorem could be the definition of concavity for such functions ($f : I \to \mathbb{R}$, $f \in C$, $f \in D$).

Using the First Derivative Test for the monotonity of f' we obtain

5.10. Theorem *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, $f \in C$, $f \in D^2$. Then*
(a) if $\forall\, x \in intI :\; f''(x) > 0$ then f is concave up;
(b) if $\forall\, x \in intI :\; f''(x) < 0$ then f is concave down .

On the graph of a differentiable function the points where the concavity changes are of special importance. These points will be called points of inflection.

5.11. Definition Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, $f \in C$, $f \in D^2$, $a \in \int I$. The point "a" is called point of inflection if exists a number $\delta > 0$ such that $(a - \delta, a + \delta \subseteq I$ and one of the following two cases holds:

Case 1: $f_{|(a-\delta,a]}$ is concave up and $f_{|[a,a+\delta)}$ is concave down
or
Case 2: $f_{|(a-\delta,a]}$ is concave down and $f_{|[a,a+\delta)}$ is concave up

Remark that the concept of point of inflection can be defined in a more general form, but for our computations the definition given above will be appropriate.

5.4. Homeworks

1. Use the L'Hospital Rule to determine the following limits:

$$a)\quad \lim_{x\to 0} \frac{\sin x - x}{\arcsin x - x} \qquad\qquad b)\quad \lim_{x\to 0} x \cdot \operatorname{ctg}(\pi x)$$

$$c)\quad \lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right) \qquad\qquad d)\quad \lim_{x\to 0+} (\cos x)^{\frac{1}{x}}$$

2. Make a complete discussion (with graphing) of the following functions:

$$a)\quad f(x) = \frac{2}{x} - \frac{3}{1+x} \qquad\qquad b)\quad f(x) = x \cdot \ln x$$

3. Let $f(x) := \sqrt{1+x}$ $(x \geq -1)$.

   a) Write the 2-nd degree Taylor's polynomial T_2 of f centered at 0.

   b) Estimate the error in the approximation $f(x) \approx T_2(x)$.

6. Lesson 6

6.1. The Antiderivative

In many cases we look for a function whose derivative is a given function. This process is called antidifferentiation or indefinite integration.

6.1. Definition Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$, $F : I \to \mathbb{R}$. The function F is called to be an antiderivative of f if $F \in D$ and

$$\forall\, x \in I : \quad F'(x) = f(x)\,.$$

About the set of antiderivatives of a function the following theorem holds:

6.2. Theorem *Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$. Let $F : I \to \mathbb{R}$ be an antiderivative of f. Then the set of all antiderivatives of f is*

$$\{I \ni x \mapsto F(x) + C \mid C \in \mathbb{R}\} = \{F + C \mid C \in \mathbb{R}\}\,.$$

Proof. Since

$$(F(x) + C)' = F'(x) + C' = f(x) + 0 = f(x)$$

therefore the function $F + C$ is indeed an antiderivative of f.

On the other hand let $G : I \to \mathbb{R}$ be an antiderivative of f. Since $G' = f$ then

$$(G - F)' = G' - F' = f - f = 0,$$

so – using the First Derivative Test – $\exists\, C \in \mathbb{R}\ \forall\, x \in I : \quad (G - F)(x) = C$. After rearrangement: $G(x) = F(x) + C$, so G is of the form $F + C$. □

6.3. Definition The set of all antiderivatives of the function f is called the indefinite integral of f and is denoted by: $\displaystyle\int f$, $\displaystyle\int f(x)\ dx$.

Remark that in the practice sometimes the individual antiderivatives are named also indefinite integral, for example:

$$\int 3x^2\ dx = x^3\,.$$

The indefinite integral in practice is written not with set notations but in the following way:

$$\int 3x^2\ dx = x^3 + C.$$

The next question that we are concerned is: Which functions have antiderivatives? Later we will prove the following theorem:

6.4. Theorem *If $I \subseteq \mathbb{R}$ is an open interval and $f : I \to \mathbb{R}$ is a continuous function then f has antiderivative.*

6.2. Five Simple Integration Rules

6.5. Theorem *[sum] Let $I \subseteq \mathbb{R}$ be open interval, $f, g : I \to \mathbb{R}$. If f and g have antiderivatives, so does $f + g$. Moreover*

$$\int f(x) + g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx\,.$$

Proof.

$$(\int f + \int g)' = (\int f)' + (\int g)' = f + g\,.$$

□

6.6. Theorem *[scalar multiple] Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$, $\lambda \in \mathbb{R}$. If f has antiderivative, so does λf, moreover*

$$\int \lambda \cdot f(x)\ dx = \lambda \cdot \int f(x)\ dx\,.$$

Proof.

$$(\lambda \cdot \int f)' = \lambda \cdot (\int f)' = \lambda \cdot f\,.$$

□

6.7. Theorem *[linear substitution] Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ and $F : I \to \mathbb{R}$ be an antiderivative of f. Furthermore let $a,\ b \in \mathbb{R}$, $a \neq 0$ and $J := \{x \in \mathbb{R} \mid ax + b \in I\}$. Then J is an open interval and*

$$\int f(ax+b)\ dx = \frac{F(ax+b)}{a} \qquad (x \in J).$$

Proof. Obviously J is an open interval. Moreover:

$$\left(\frac{F(ax+b)}{a}\right)' = \frac{1}{a} \cdot F'(ax+b) \cdot a = f(ax+b). \quad (x \in J)\,.$$

□

6.8. Theorem *[integrals of type $f^\alpha \cdot f'$] Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ be continuous, $\alpha \in \mathbb{R}$. Suppose that the power $(f(x))^\alpha$ is defined for every $x \in I$. Then*

a) if $\alpha \neq -1$ then

$$\int (f(x))^\alpha \cdot f'(x)\ dx = \frac{(f(x))^{\alpha+1}}{\alpha+1} \qquad (x \in I);$$

b) if $\alpha = -1$ *then*

$$\int (f(x))^{-1} \cdot f'(x)\,dx = \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| \qquad (x \in I).$$

Proof.

a) If $\alpha \neq -1$ then:

$$\left(\frac{f^{\alpha+1}}{\alpha+1}\right)' = \frac{1}{\alpha+1} \cdot (\alpha+1) \cdot f^{\alpha} \cdot f' = f^{\alpha} \cdot f';$$

b) If $\alpha = -1$ then:

$$(\ln|f|)' = \frac{1}{f} \cdot f' = \frac{f'}{f}.$$

□

6.3. Integration of Rational Functions

Using the previous simple integration rules we can give a method for integration of rational functions. Recall that a rational function is a quotient of two polynomials.

Let us study important basic types first:

Basic type 1:

Let $A \in \mathbb{R},\ \alpha \in \mathbb{R},\ k \in \mathbb{N}^+,\ I := (-\infty,\ \alpha)$ or $I := (\alpha,\ +\infty)$ and

$$R(x) := \frac{A}{(x-\alpha)^k} \qquad (x \in I)\,.$$

In this case – using the rule of the linear substitution – we obtain

$$\int R(x)\,dx = \int \frac{A}{(x-\alpha)^k}\,dx = \begin{cases} \dfrac{A \cdot (x-\alpha)^{-k+1}}{-k+1}, & \text{if k}\geq 2; \\ A \cdot \ln|x-\alpha|, & \text{if k=1.} \end{cases}$$

Basic type 2:

Let $B,\, C \in \mathbb{R},\ \beta,\, \gamma \in \mathbb{R}$ where $\beta^2 - 4\gamma < 0,\ I := \mathbb{R}$ and

$$R(x) := \frac{Bx + C}{x^2 + \beta x + \gamma} \qquad (x \in I)\,.$$

In this case we apply the following method. If $B \neq 0$ then we introduce the derivative of the denominator into the numerator:

$$\frac{Bx+C}{x^2+\beta x+\gamma} = \frac{B}{2} \cdot \frac{2x+\dfrac{2C}{B}}{x^2+\beta x+\gamma} = \frac{B}{2} \cdot \frac{2x+\beta+\dfrac{2C}{B}-\beta}{x^2+\beta x+\gamma} =$$
$$= \frac{B}{2} \cdot \frac{2x+\beta}{x^2+\beta x+\gamma} + \frac{B}{2} \cdot (\frac{2C}{B}-\beta) \cdot \frac{1}{x^2+\beta x+\gamma}$$

The first term is $\dfrac{f'}{f}$ type so its integration is easy. The second term is like the original function R but in the numerator the constant 1 stands instead of the linear function $Bx+C$.

If originally $B=0$ then after separating the factor C we obtain the above fraction.

So the problem is reduced to the form

$$\frac{1}{x^2+\beta x+\gamma}\,.$$

The integration of this fraction will be solved by „eliminating" the term βx (if $\beta \neq 0$) in the denominator by „completing the square". After this step we will transform the obtained fraction into the form „$\dfrac{1}{(ax+b)^2+1}$" which – using the linear substitution and the antiderivative of arc tg – can be integrated easily:

$$\frac{1}{x^2+\beta x+\gamma} = \frac{1}{\left(x+\dfrac{\beta}{2}\right)^2 - \dfrac{\beta^2}{4}+\gamma} = \frac{1}{\gamma-\dfrac{\beta^2}{4}} \cdot \frac{1}{\left(\dfrac{x+\dfrac{\beta}{2}}{\sqrt{\gamma-\dfrac{\beta^2}{4}}}\right)^2+1} =$$
$$= \frac{4}{4\gamma-\beta^2} \cdot \frac{1}{\left(\dfrac{2}{\sqrt{4\gamma-\beta^2}}x+\dfrac{\beta}{\sqrt{4\gamma-\beta^2}}\right)^2+1}$$

Basic type 3:

Let $B,\, C \in \mathbb{R},\ \beta,\, \gamma \in \mathbb{R}$ where $\beta^2-4\gamma<0,\ k \in \mathbb{N},\ k \geq 2,\ I := \mathbb{R}$ and

$$R(x) := \frac{Bx+C}{(x^2+\beta x+\gamma)^k} \qquad (x \in I)\,.$$

In this case – using a recursive process – we trace the integral of R back to a similar integral but with exponent $k-1$ instead of k. The recursive process is continued until the exponent k will be reduced to 1 and the problem becomes of Basic type 2.

The recursion process is based on the following theorem:

6.9. Theorem *There exist constants B_1, C_1, $D_1 \in \mathbb{R}$ such that:*

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx = \frac{B_1x+C_1}{(x^2+\beta x+\gamma)^{k-1}} + \int \frac{D_1}{(x^2+\beta x+\gamma)^{k-1}}\,dx.$$

Arbitrary rational functions:

An arbitrary rational function can be integrated by the method of partial fraction decomposition.

6.10. Theorem *Let P and Q be nonzero real polynomials where the root factor form of Q over $\mathbb{R}$ is:*

$$Q(x) = (x-\alpha_1)^{m_1}\cdot\ldots\cdot(x-\alpha_r)^{m_r}\cdot(x^2+\beta_1x+\gamma_1)^{n_1}\cdot\ldots\cdot(x^2+\beta_sx+\gamma_s)^{n_s}\,.$$

Here $\alpha_1,\ldots\alpha_r$ are the real roots of Q with multiplicities $m_1,\ldots,m_r$, $\beta_i^2-4\gamma_i<0$ $(j=1,\ldots s)$, $m_1+\ldots+m_r+2n_1+\ldots+2n_s=\deg Q$. Then R can be written in form

$$R(x) := \frac{P(x)}{Q(x)} = S(x) + \sum_{i=1}^{r}\sum_{j=1}^{m_i}\frac{A_{ij}}{(x-\alpha_i)^j} + \sum_{i=1}^{s}\sum_{j=1}^{n_i}\frac{B_{ij}x+C_{ij}}{(x^2+\beta_ix+\gamma_i)^j},$$

where S is a polynomial, A_{ij}, B_{ij}, C_{ij} are real coefficients.

So the integral of a rational fraction is the sum of the integral of a polynomial and of the integrals of some basic type rational fractions.

6.4. Homeworks

1. Determine the antiderivatives

a) $\displaystyle\int \frac{5}{\cos^2(-6x+4)}\,dx$ *b)* $\displaystyle\int \frac{2x-5}{\sqrt[3]{(x^2-5x+13)^7}}\,dx$

c) $\displaystyle\int \frac{2x^2-5}{(x-2)(x^2-1)}\,dx$ *d)* $\displaystyle\int \frac{6x}{x^2-2x+17}\,dx$

e) $\displaystyle\int \frac{x^2}{(x-1)(x^2+2x+1)}\,dx$

7. Lesson 7

7.1. Integration by Parts

7.1. Theorem *[integration by parts] Let $I \subseteq \mathbb{R}$ be an open interval, $f, g : I \to \mathbb{R}$, $f, g, \in D$, f', $g' \in C$. Then*

$$\int f(x) \cdot g'(x)\ dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\ dx \qquad (x \in I)$$

Proof. Apply the product rule of derivative:

$$(f \cdot g - \int (f' \cdot g))' = (f \cdot g)' - (\int (f' \cdot g))' = f' \cdot g + f \cdot g' - f' \cdot g = f \cdot g'\,.$$

□

7.2. Substitution

7.2. Theorem *[Substitution, form I.] Let I, $J \subseteq \mathbb{R}$ be open intervals, $f : J \to \mathbb{R}$, $f \in C$, $g : I \to J$ $g \in D$, $g' \in C$. Then*

$$\int (f \circ g) \cdot g' = (\int f) \circ g.$$

Proof. Apply the Chain Rule:

$$((\int f) \circ g)' = ((\int f)' \circ g) \cdot g' = (f \circ g) \cdot g'\,.$$

□

7.3. Remark. If the variable of g is denoted by x then the rule of substitution can be written as:

$$\int f(g(x)) \cdot g'(x)\ dx = F(g(x)) \quad (x \in I)\,,$$

where F denotes an antiderivative of f. So – denoting the variable of f by u–:

$$F(u) = \int f(u)\ du \quad (u \in J)\,.$$

From here can we obtain the practical process of the substitution. In the integral $\int f(g(x)) \cdot g'(x)\ dx$ substitute $g(x)$ by u and $g'(x)\ dx$ by du. After determination of this new integral substitute u by $g(x)$.

Now suppose that g is a bijection (naturally because of its continuity it is strictly monotone too). Then the above rule of substitution can be used in another form:

7.4. Theorem *[Substitution, form II.] Let $I,\ J \subseteq \mathbb{R}$ be open intervals, $f : I \to \mathbb{R}$, $f \in C$, $g : J \to I$, $g \in D$, $g' \in C$. Then*

$$\int f = \left(\int (f \circ g) \cdot g' \right) \circ g^{-1}.$$

Proof. In the substitution formula (form I.) let us interchange the roles of I and J:

$$\int (f \circ g) \cdot g' = (\int f) \circ g.$$

Then take the composition of both sides with the function $g^{-1} : I \to J$:

$$\left(\int (f \circ g) \cdot g' \right) \circ g^{-1} = (\int f) \circ g \circ g^{-1} = \int f.$$

After interchange the sides of this equality we obtain the desired formula. □

7.5. Remarks. 1. If the variable of f is denoted by x, the variable of g is denoted by t then the form II. can be written as:

$$\int f(x)\ dx = \int f(g(t)) \cdot g'(t)\ dt\,|_{t=g^{-1}(x)} \qquad (x \in I).$$

From here comes the practical process of substitution (form II.). In the integral $\int f(x)\ dx$ x is replaced by $g(t)$, dx is replaced by $g'(t)\ dt$. After computation of this integral replace t by $g^{-1}(x)$.

2. The function g can be chosen freely. There are „suggested" substitutions for many types of integral,

3. If the conditions of the theorem about the form II. hold then both forms of the substitution can be used.

7.3. Homeworks

1. Determine the following integrals:

$$\int x^2 \cdot \ln x\ dx \qquad \int (x^2+1) \cdot e^{2x}\ dx \qquad \int x \cdot \cos x\ dx$$

2. Determine the following integrals:

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\ dx \qquad \int \frac{\sqrt{9-4x^2}}{x}\ dx \qquad \int \frac{x}{\sqrt{3x+5}}\ dx$$

8. Lesson 8

8.1. The definite Integral

8.1. Definition Let $n \in \mathbb{N}$ and divide the interval $[a,\, b]$ into n closed subintervals $[x_{i-1},\, x_i]$ $(i = 1, \ldots, n)$ where:

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b\,.$$

The finite set $\{x_0, x_1, \ldots, x_n\}$ is called a partition of the interval $[a,\, b]$.

The set of the partitions of $[a,\, b]$ will be denoted by $\mathcal{P}[a,\, b]$.

8.2. Definition Let $P \in \mathcal{P}[a,\, b]$ Then the length of the longest subinterval is called the norm of the partition P:

$$||P|| := \max\{x_i - x_{i-1} \mid i = 1, \ldots, n\}\,.$$

It is obvious that for every $\delta > 0$ there exists a partition P „finer" than δ that is $||P|| < \delta$.

8.3. Definition Let $f : [a,\, b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, \ldots, x_n\} \in \mathcal{P}[a,\, b]$. Let

$$m_i := \inf\{f(x) \mid x_{i-1} \le x \le x_i\}, \qquad M_i := \sup\{f(x) \mid x_{i-1} \le x \le x_i\} \qquad (i = 1, \ldots, n)\,.$$

We introduce the following sums:

a) lower sum: $s(f,\, P) := \sum\limits_{i=1}^{n} m_i \cdot (x_i - x_{i-1})$,

b) upper sum: $S(f,\, P) := \sum\limits_{i=1}^{n} M_i \cdot (x_i - x_{i-1})$.

8.4. Theorem *If $P,\, Q \in \mathcal{P}[a,\, b]$, $P \subseteq Q$ then $s(f,\, P) \le s(f,\, Q)$ and $S(f,\, P) \ge S(f,\, Q)$.*

8.5. Corollary. If $P,\, Q \in \mathcal{P}[a,\, b]$ then

$$s(f,\, P) \le s(f,\, P \cup Q) \le S(f,\, P \cup Q) \le S(f,\, Q)\,.$$

8.6. Corollary. The set of the lower sums is bounded above, the set of the upper sums is bounded below.

8.7. Definition The number $I_*(f) := \sup\{s(f,\, P) \mid P \in \mathcal{P}[a,\, b]\}$ is called the lower integral of f. Respectively the number $I^*(f) := \inf\{S(f,\, P) \mid P \in \mathcal{P}[a,\, b]\}$ is called the upper integral of f.

8.8. Definition A function $f : [a,b] \to \mathbb{R}$ is called to be integrable if it is bounded and $I_*(f) = I^*(f)$. This common value of the lower and upper integral is called the integral of f from a to b and is denoted by

$$\int_a^b f, \quad \int_a^b f(x)\, dx\,.$$

In this connection the number a is called the lower limit of the integral and the number b is called the upper limit of the integral.

The definition can be extended easily to the case when the domain of f is wider than $[a,b]$:

8.9. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $[a,b] \subseteq D_f$. We say that f is integrable over the interval $[a,b]$ if the restricted function $f_{\big|[a,b]}$ is integrable. The integral of f from a to b is denoted by the previous way and is defined as

$$\int_a^b f := \int_a^b f(x)\, dx := \int_a^b f_{\big|[a,b]}\,.$$

The set of integrable functions over $[a,b]$ is denoted by $R[a,b]$.

From the definition it follows that in the case $f(x) \geq 0$ $(x \in [a,b])$ the geometrical meaning of the integral is the area of the planar region „under the graph of f" that is the area of the region

$$R := \left\{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ 0 \leq y \leq f(x)\right\}\,.$$

8.10. Examples

1. Let $c \in \mathbb{R}$ be fixed and $f(x) := c$ $(x \in [a,b])$ be the constant function. Then for any partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,\, b]$:

   $$s(f,\, P) := \sum_{i=1}^{n} c \cdot (x_i - x_{i-1}) = c \cdot \sum_{i=1}^{n} (x_i - x_{i-1}) = c \cdot (b-a)\,,$$

   which implies that $I_*(f) = c \cdot (b-a)$.

   On the other hand

   $$S(f,\, P) := \sum_{i=1}^{n} c \cdot (x_i - x_{i-1}) = c \cdot \sum_{i=1}^{n} (x_i - x_{i-1}) = c \cdot (b-a)\,,$$

   which implies that $I^*(f) = c \cdot (b-a)$.

   So

   $$\int_a^b f(x)\, dx = I_*(f) = I^*(f) = c \cdot (b-a)\,.$$

2. Here follows an example for nonintegrable (but bounded) function.

   Let $f : [a,b] \to \mathbb{R}$ be the following function:

$$f(x) := \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \cap [a,b] \\ \\ 0 & \text{if} \quad x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [a,b]. \end{cases}$$

   Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. Since every subinterval $[x_{i-1},\, x_i]$ contains rational and irrational numbers too we have

$$m_i := \inf\{f(x) \mid x_{i-1} \le x \le x_i\} = 0,$$

$$M_i := \sup\{f(x) \mid x_{i-1} \le x \le x_i\} = 1 \qquad (i = 1, \dots, n)\,.$$

   Thus

$$s(f,\, P) := \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1}) = 0, \qquad S(f,\, P) := \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1}) = b - a\,.$$

   Consequently

$$I_*(f) = \sup_P s(f,\, P) = 0, \qquad I^*(f) = \inf\{S(f,\, P) = b - a\,.$$

   They are not equal, so $f \notin R[a,b]$.

8.2. Oscillation Sum

8.11. Definition Let $f : [a,\, b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, \dots, x_n\} \in \mathcal{P}[a,\, b]$. The number

$$\Omega(f,P) := S(f,P) - s(f,P) = \sum_{i=1}^{n} (M_i - m_i) \cdot (x_i - x_{i-1})$$

is called oscillation sum. (M_i, m_i were defined in the previous section.)

The following theorem will be useful when we want to prove the integrability of a function.

8.12. Theorem *Let $f : [a,\, b] \to \mathbb{R}$ be a bounded function. Then*

$$f \in R[a,b] \Leftrightarrow \forall\, \varepsilon > 0\; \exists\, P \in \mathcal{P}[a,b] : \quad \Omega(f,P) < \varepsilon\,.$$

Proof. Assume that $f \in R[a,b]$ and let $\varepsilon > 0$. By the definition of the least upper bound

$$\exists\, P_1 \in \mathcal{P}[a,b]: \quad s(f,P_1) > I_*(f) - \frac{\varepsilon}{2}\,.$$

Similarly by the definition of the greatest lower bound

$$\exists\, P_2 \in \mathcal{P}[a,b]: \quad S(f,P_2) < I^*(f) + \frac{\varepsilon}{2}\,.$$

So we can write with $P := P_1 \cup P_2$:

$$\Omega(f,P) := S(f,P) - s(f,P) \le S(f,P_2) - s(f,P_1) < I^*(f) + \frac{\varepsilon}{2} - \Big(I_*(f) - \frac{\varepsilon}{2}\Big) = \varepsilon\,.$$

Conversely let $\varepsilon > 0$ be an arbitrary but fixed positive number and P be a partition with $\Omega(f,P) < \varepsilon$. Then

$$0 \le I^*(f) - I_*(f) \le S(f,P) - s(f,P) = \Omega(f,P) < \varepsilon$$

Since it is true for any $\varepsilon > 0$ we infer that $I_*(f)$ is equal to $I^*(f)$. □

8.3. „Backward” integration

It is convenient and useful to extend the integration if its lower limit is greater than or equal to its upper limit.

8.13. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $f \in R[a,b]$. Then

$$\int_b^a f(x)\,dx := -\int_a^b f(x)\,dx\,.$$

8.14. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in D_f$. Then

$$\int_a^a f(x)\,dx := 0\,.$$

So we have defined the definite integral $\int_a^b f(x)\,dx$ for any pair $a,\, b \in \mathbb{R}$. For any pair $a,\, b \in \mathbb{R}$ let us denote the set of functions for which the integral $\int_a^b f(x)\,dx$ exists (independently of $a < b$, $a = b$, $a > b$) by $R[a,b]$.

8.4. The properties of the definite integral

In this section the theorems are stated without proofs.

8.15. Theorem *[Addition] Let $a,\, b \in \mathbb{R}$, $f, g \in R[a,b]$. Then $f+g \in R[a,b]$ and* $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

8.16. Theorem *[Constant Multiple] Let $a,\, b \in \mathbb{R}$, $f \in R[a,b]$, $c \in \mathbb{R}$. Then $cf \in R[a,b]$ and* $\int_a^b cf = c \cdot \int_a^b f$.

8.17. Theorem *[Interval Additivity] Let $a,\, b,\, c \in \mathbb{R}$ and put them in nondecreasing order: $A \le B \le C$. Then*

$$f \in R[A,C] \quad \Leftrightarrow \quad f \in R[A,B] \quad \text{and} \quad f \in R[B,C]\,.$$

In this case:

$$\int_a^c f = \int_a^b f + \int_b^c f\,.$$

8.18. Corollary. If $a < b$ and $f \in R[a,b]$ then for every $[c,d] \subseteq [a,b]$: $f \in R[c,d]$.

In the following theorems $a < b$ is assumed.

8.19. Theorem *[Monotonity] Let $a,\, b \in \mathbb{R}$, $a < b$, $f, g \in R[a,b]$. Suppose that $f(x) \le g(x)$ $(x \in [a,b])$. Then*

$$\int_a^b f \le \int_a^b g\,.$$

8.20. Theorem *[„Triangle" inequality] Let $a,\, b \in \mathbb{R}$, $a < b$, $f \in R[a,b]$. Then $|f| \in R[a,b]$ and*

$$\left| \int_a^b f \right| \le \int_a^b |f|\,.$$

8.21. Theorem *[Mean value Theorem]*

Let $a,\, b \in \mathbb{R}$, $a < b$, $f, g \in R[a,b]$, $g(x) \ge 0$ $(x \in [a,b])$. Let

$$m := \inf\{f(x) \mid a \le x \le b\}, \qquad M := \sup\{f(x) \mid a \le x \le b\}\,.$$

Then

$$m \cdot \int_a^b g \le \int_a^b fg \le M \cdot \int_a^b g\,.$$

Moreover if f is continuous on $[a,b]$ then

$$\exists\, \xi \in [a,b] : \qquad \int_a^b fg = f(\xi) \cdot \int_a^b g\,.$$

8.22. Theorem *[Change of the function at a point] Let $a,\, b \in \mathbb{R}$, $a < b$, $f \in R[a,b]$. Let $x_0 \in [a,b]$, $k \in \mathbb{R}$ and*

$$g(x) := \begin{cases} f(x) & \textit{if} \quad x \in [a,b] \setminus \{x_0\} \\ \\ k & \textit{if} \quad x = x_0. \end{cases}$$

Then $g \in R[a,b]$ and $\int_a^b f = \int_a^b g$.

8.23. Corollary. Let $a,\, b \in \mathbb{R}$, $a < b$, $f \in R[a,b]$. If the function $g : [a,b] \to \mathbb{R}$ differs from $f_{\big|[a,b]}$ only on a finite subset of $[a,b]$ then $g \in R[a,b]$ and $\int_a^b f = \int_a^b g$. It follows by applying the previous theorem finitely many times.

This theorem makes us possible to give a generalization of the integral for functions that are not defined at a finite number of points of the interval.

8.24. Definition Let $H = \{h_1,\, \dots\, h_n\} \subseteq [a,b]$ be a finite set, $f : [a,b] \setminus H \to \mathbb{R}$ be a function, $c_1,\, \dots\, c_n \in \mathbb{R}$. We say that f is integrable if for the function

$$g(x) := \begin{cases} f(x) & \text{if} \quad x \in [a,b] \setminus H \\ \\ c_i & \text{if} \quad x \in H, x = h_i. \end{cases}$$

holds $g \in R[a,b]$. In this case

$$\int_a^b f(x)\; dx := \int_a^b g(x)\; dx\,.$$

By the previous theorem and its corollary the definition is independent of choosing $c_1,\, \dots\, c_n$.

We say shortly that f is integrable if it has an integrable extension to $[a,b]$.

Notation: in the special case $H = \{a,\, b\}$ let us denote by $R(a,b)$ the set of functions $f : (a,b) \to \mathbb{R}$ which are integrable (in the previous sense).

8.5. Homeworks

1. Let $f : [a,b] \to \mathbb{R}$ and $P_n = \{x_0^{(n)}, x_1^{(n)}, \ldots, x_n^{(n)}\} \in \mathcal{P}[a,\, b]$ $(n \in \mathbb{N})$ a sequence of partitions. Suppose that

$$\lim_{n\to\infty} s(f,\, P_n) = \lim_{n\to\infty} S(f,\, P_n) = I \in \mathbb{R}\,.$$

Then $f \in R[a,b]$ and $\int\limits_a^b f(x)\ dx = I$.

2. Prove – using the previous exercise – that $f(x) = x^2$ is integrable over $[0,\, 1]$ and compute its integral.

9. Lesson 9

9.1. Integrability of continuous functions

9.1. Theorem *Let $f : [a,\, b] \to \mathbb{R}$ be a continuous function. Then f is integrable.*

Proof. The essence of the proof is to give to any $\varepsilon > 0$ a partition P such that $\Omega(f, P) < \varepsilon$.

So let $\varepsilon > 0$ be fixed and let us construct P as follows.

By Heine's theorem (Theorem 2.6) f is uniformly continuous. Then there exists a $\delta > 0$ such that

$$\forall\, s,\, t \in [a, b],\ |s - t| < \delta : \quad |f(s) - f(t)| < \frac{\varepsilon}{b - a}\,.$$

Let $P = \{x_0, x_1, \ldots, x_n\} \in \mathcal{P}[a, b]$ be a partition „finer" than δ, that is $||P|| < \delta$.

Using Weierstrass's theorem (Theorem 2.3)

$$\exists\, \xi_i,\, \eta_i \in [x_{i-1},\, x_i] : \quad m_i = f(\xi_i),\ M_i = f(\eta_i) \qquad (i = 1, \ldots n)\,.$$

We want to apply the uniform continuity of f with $s := \xi_i$, $t := \eta_i$. We check first that

$$|s - t| = |\xi_i - \eta_i| \le x_i - x_{i-1} \le ||P|| < \delta\,,$$

so the uniform continuity can be applicable:

$$M_i - m_i = f(\xi_i) - f(\eta_i) = |f(\xi_i) - f(\eta_i)| < \frac{\varepsilon}{b - a} \qquad (i = 1, \ldots n)\,.$$

Consequently,

$$\Omega(f, P) = \sum_{i=1}^{n} (M_i - m_i) \cdot (x_i - x_{i-1}) < \sum_{i=1}^{n} \frac{\varepsilon}{b - a} \cdot (x_i - x_{i-1})$$
$$= \frac{\varepsilon}{b - a} \cdot \sum_{i=1}^{n} (x_i - x_{i-1}) = \frac{\varepsilon}{b - a} \cdot (b - a) = \varepsilon\,.$$

□

9.2. Piecewise continuous functions

9.2. Definition Let $[a, b] \subseteq \mathbb{R}$ be a closed bounded interval and $f \in [a, b] \to \mathbb{R}$. We say that f is piecewise continuous if there is a partition $P = \{x_0, \ldots x_n\}$ of $[a, b]$ such that

1. The functions $f\big|_{(x_{i-1},\, x_i)}$ (i=1, … n) are continuous;
2. The left-hand side limits at the points $x_1, \dots x_n$ exist and are finite;
3. The right-hand side limits at the points $x_0, \dots x_{n-1}$ exist and are finite.

Remark that by this definition every continuous function is piecewise continuous, and that a piecewise continuous function may be undefined at most at a finite number of points of $[a, b]$. So it is sensible to ask that a piecewise continuous function is integrable or not.

9.3. Theorem *Let $f \in [a, b] \to \mathbb{R}$ be a piecewise continuous function. Then it is integrable and*

$$\int_a^b f(x)\, dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\, dx\,,$$

where $P = \{x_0, \dots x_n\}$ is a partition given in the definition above. It is easy to see that the integrability and the value of integral is independent of the particular partition.

Proof. Let $c_0, \dots c_n \in \mathbb{R}$ and denote by f the extension of f to $[a, b]$ by $f(x_i) := c_i$ too. Then for a fixed i $f\big|_{[x_{i-1},\, x_i]}$ differs from the following integrable function g_i at most at x_{i-1} or x_i:

$$g_i(x) := \begin{cases} f(x) & \text{if} \quad x \in (x_{i-1}, x_i) \\ \lim\limits_{x \to x_i - 0} & \text{if} \quad x = x_i \;\; i = 1, \dots n \\ \lim\limits_{x \to x_i + 0} & \text{if} \quad x = x_i \;\; i = 0, \dots n-1 \end{cases}$$

Obviously g_i is continuous, so $g_i \in R[x_{i-1},\, x_i]$. Consequently, for f holds $f \in R[x_{i-1},\, x_i]$. Finally using the Interval Additivity Theorem follows that $f \in R[a, b]$ and

$$\int_a^b f(x)\, dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\, dx\,.$$

□

9.3. Integral Function

In this section we discuss the definite integral as a function of its upper limit.

9.4. Definition Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ a function and suppose that f is integrable over every closed and bounded subinterval of I (this is the case for example if f is continuous). Fix a point $a \in I$. The function

$$F : I \to \mathbb{R}, \quad F(x) := \int_a^x f(t)\, dt \qquad (x \in I)$$

is called the integral function of f (vanishing at a).

The property „vanishing at a" expresses the triviality $F(a) = 0$.

9.5. Theorem *[the continuity of the Integral Function] Using the notations of the previous definition:*

$F : I \to \mathbb{R}$ is continuous.

Proof. Since every point of I can be covered by a closed bounded subinterval of I, it is enough to prove that for every closed and bounded subinterval $[\alpha, \beta] \subseteq I$ $F\big|_{[\alpha,\beta]}$ is uniformly continuous. Fix such an interval $[\alpha, \beta]$. Since f is integrable on $[\alpha, \beta]$, $f\big|_{[\alpha,\beta]}$ is bounded. Denote by M a bound of it:

$$|f(x)| \le M \qquad (x \in [\alpha, \beta])\,.$$

Then for every $x, y \in [\alpha, \beta]$, $x < y$ we can write:

$$|F(x) - F(y)| = \left|\int_a^x f(t)\, dt - \int_a^y f(t)\, dt\right| = \left|\int_y^x f(t)\, dt\right| = \left|\int_x^y f(t)\, dt\right| \le$$

$$\le \int_x^y |f(t)|\, dt \le \int_x^y M\, dt = M \cdot (y - x) = M \cdot |x - y|\,.$$

This implies the uniform continuity of $F\big|_{[\alpha,\beta]}$. □

9.6. Theorem *[the right-hand differentiability of the Integral Function]*

Using the notations of the definition of the Integral Function:

Suppose that $x \in I$ but x is not the right endpoint of I. Suppose that f is „continuous from the right" at x that is

$$\lim_{h \to 0+} f(x + h) = f(x)\,.$$

Then

$$F'_+(x) := \lim_{h \to 0+} \frac{F(x + h - F(x)}{h} = f(x)\,.$$

In words: F is differentiable from the right and its right-hand derivative equals $f(x)$.

Proof. Let $\varepsilon > 0$. Then for every $h > 0$ with $x + h \in I$:

$$\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| = \frac{1}{h}\cdot |F(x+h)-F(x)-f(x)\cdot h| =$$

$$= \frac{1}{h}\cdot\left|\int_a^{x+h} f(t)\ dt - \int_a^{x} f(t)\ dt - \int_x^{x+h} f(x)\ dt\right| =$$

$$= \frac{1}{h}\cdot\left|\int_x^{x+h} f(t)dt - \int_x^{x+h} f(x)\ dt\right| = \frac{1}{h}\cdot\left|\int_x^{x+h} (f(t)-f(x))\ dt\right| \leq \frac{1}{h}\cdot\int_x^{x+h} |f(t)-f(x)|\ dt\,.$$

Since f is continuous from the right at x we have that

$$\exists\,\delta > 0, (x, x+\delta) \subset I\ \forall\, t \in (x, x+\delta):\quad |f(t)-f(x)| < \varepsilon\,.$$

Let $0 < h < \delta$. Then – because of $(x, x+h) \subset (x, x+\delta)$:

$$\forall\, t \in (x, x+h):\quad |f(t)-f(x)| < \varepsilon\,.$$

So we can continue the above estimation by

$$\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| \leq \frac{1}{h}\cdot\int_x^{x+h} |f(t)-f(x)|\ dt \leq \frac{1}{h}\cdot\int_x^{x+h} \varepsilon\ dt = \frac{1}{h}\cdot\varepsilon\cdot h = \varepsilon.$$

We have proved that

$$\forall\,\varepsilon > 0\ \exists\,\delta > 0, (x, x+\delta) \subset I\ \forall\, h \in (x, x+\delta):\quad \left|\frac{F(x+h)-F(x)}{h} - f(x)\right| < \varepsilon.$$

This means exactly the statement of the theorem. □

A similar theorem can be proved for the left-hand derivative:

9.7. Theorem *[the left-hand differentiability of the Integral Function] Using the notations of the definition of the Integral Function:*

Suppose that $x \in I$ but x is not the left endpoint of I. Suppose that f is „continuous from the left” at x that is

$$\lim_{h\to 0-} f(x+h) = f(x)\,.$$

Then

$$F'_-(x) := \lim_{h\to 0-} \frac{F(x+h-F(x)}{h} = f(x)\,.$$

In words: F is differentiable from the left and its left-hand derivative equals $f(x)$.

9.8. Corollary. If $x \in intI$ and $f \in C(x)$ then F is differentiable at x and $F'(x) = f(x)$.

9.9. Corollary. If $I \subseteq \mathbb{R}$ is an open interval and $f : I \to \mathbb{R}$ is continuous then f has an antiderivative. So we have proved the 6.4 Theorem.

10. Lesson 10

10.1. The Fundemental Theorem of Calculus (Newton-Leibniz)

10.1. Theorem *[Newton-Leibniz's Formula] Let $[a, b] \subseteq \mathbb{R}$ be a closed bounded interval and $f \in R[a, b]$. Suppose that there exists a function $F : [a, b] \to \mathbb{R}$ for which:*

1. *F is continuous on $[a, b]$;*
2. *F is differentiable on (a, b);*
3. *$F'(x) = f(x) \quad (x \in (a, b))$.*

Then

$$\int_a^b f(x)\, dx = F(b) - F(a)\,.$$

Proof. Let $P = \{x_0,\, \dots\, x_n\} \in \mathcal{P}[a, b]$ be a partition. Apply for F the Lagrange Mean Value Theorem on the i-th subinterval $[x_{i-1},\, x_i]$ $(i = 1,\, \dots\, n)$:

$$\exists\, \xi_i \in (x_{i-1},\, x_i) : \quad F(x_i) - F(x_{i-1} = F'(\xi_i) \cdot (x_i - x_{i-1}) = f(\xi_i) \cdot (x_i - x_{i-1})\,.$$

Using this fact we can write

$$F(b) - F(a) = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1})\,.$$

By the usual notations we have for every $i = 1, \dots, n$:

$$m_i = \inf\{f(x) \mid x_{i-1} \le x \le x_i\}, \qquad M_i = \sup\{f(x) \mid x_{i-1} \le x \le x_i\}\,.$$

It is clear that $m_i \le f(\xi_i) \le M_i$, so

$$s(f,\, P) = \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1}) \le \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1}) \le \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1}) = S(f,\, P)\,.$$

This means:

$$s(f,\, P) \le F(b) - F(a) \le S(f,\, P)$$

for every partition. Consequently

$$I_*(f) := \sup\{s(f,\, P) \mid P \in \mathcal{P}[a,\, b]\} \le F(b) - F(a) \le$$
$$\le \inf\{S(f,\, P) \mid P \in \mathcal{P}[a,\, b]\} = I^*(f)\,.$$

Since $f \in R[a,b]$ both $I_*(f)$ and $I^*(f)$ are equal to $\int\limits_a^b f(x)\,dx$, and so to $F(b) - F(a)$. Thus the proof is complete. □

10.2. Remarks. 1. The difference $F(b) - F(a)$ often is denoted by $[F(x)]_a^b$ or $F(x)\big|_a^b$.

2. Frequently in the applications $I \subseteq \mathbb{R}$ is an open interval , $f : I \to \mathbb{R}$ is a continuous function, and $[a,b] \subset I$. Since every continuous function is integrable and has antiderivative, the assumptions of the Newton-Leibniz's formula obviously hold.

3. The Newton-Leibniz's formula is valid for the „backward" integral too. Indeed, if the assumptions of the theorem hold, then

$$\int\limits_b^a f = -\int\limits_a^b f = -\left(F(b) - F(a)\right) = F(a) - F(b) = [F(x)]_b^a$$

and

$$\int\limits_a^a f = 0 = F(a) - F(a) = [F(x)]_a^a \,.$$

10.2. Integration by Parts

10.3. Theorem *[Integration by Parts] Let $I \subseteq \mathbb{R}$ be an open interval, $f, g : I \to \mathbb{R}$, $f, g \in D$, $f', g' \in C$. Then for every closed and bounded subinterval $[a,b] \subset I$ holds*

$$\int\limits_a^b f(x) \cdot g'(x)\,dx = [f(x) \cdot g(x)]_a^b - \int\limits_a^b f'(x) \cdot g(x)\,dx \,.$$

Proof. Apply the Newton-Leibniz formula and the partial integration rule for indefinite integrals:

$$\int\limits_a^b f(x) \cdot g'(x)\,dx = \left[\int f(x) \cdot g'(x)\,dx\right]_a^b = \left[f(x) \cdot g(x) - \int f'(x) \cdot g(x)\,dx\right]_a^b =$$

$$= f(b)g(b) - \left(\int f'(x)g(x)\,dx\right)_{|x=b} - f(a)g(a) + \left(\int f'(x)g(x)\,dx\right)_{|x=a} =$$

$$= [f(x)g(x)]_a^b - \left[\int f'(x)g(x)\,dx\right]_a^b = [f(x)g(x)]_a^b - \int\limits_a^b f'(x)g(x)\,dx \,.$$

□

10.4. Remark. It is easy to see that the partial integration rule is valid for the „backward” integral too.

10.3. Substitution

10.5. Theorem *[Substitution, Form I.] Let I, $J \subseteq \mathbb{R}$ be open intervals, $g : I \to J$, $g \in D$, $g' \in C$. Further let $f : J \to \mathbb{R}$ be a continuous function. Then for every closed and bounded subinterval $[a, b] \subset I$ we have*

$$\int_a^b f(g(x)) \cdot g'(x)\ dx = \int_{g(a)}^{g(b)} f(u) du\,.$$

Remark that the integral on the right side may be "backward".

Proof. Apply the Newton-Leibniz formula and the substitution (form I.) rule for indefinite integrals:

$$\int_a^b f(g(x)) \cdot g'(x)\ dx = \left[\int f(g(x)) \cdot g'(x)\ dx\right]_a^b = \left[(\int f(u)\ du)_{|u=g(x)}\right]_a^b =$$

$$= (\int f(u)\ du)_{|u=g(b)} - (\int f(u)\ du)_{|u=g(a)} == \left[(\int f(u)\ du)\right]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u) du\,.$$

□

The substitution formula of form II. for definite integrals can be proved similarly:

10.6. Theorem *[Substitution, Form II.] Let I, $J \subseteq \mathbb{R}$ be open intervals, $g : I \to J$, $g \in D$, $g' \in C$, g is strictly monotone. Further let $f : J \to \mathbb{R}$ be a continuous function. Then for every closed and bounded subinterval $[a, b] \subset J$ we have*

$$\int_a^b f(x)\ dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t)) \cdot g'(t)\ dt\,.$$

Remark that the integral on the right side may be "backward".

10.7. Remark. It is easy to see that the substitution rules are valid for the „backward” integral too.

10.4. Homeworks

1. Determine the following definite integrals:

$$\int_0^{\pi/2} \sin(3x)\ dx \qquad \int_{-1}^{1} \frac{1}{2x-4}\ dx \qquad \int_2^{3} \frac{1}{(2x-1)^3}\ dx \qquad \int_0^{4} \sqrt{3x+4}\ dx$$

2. Determine the following definite integrals:

$$\int_1^{e} x^2 \cdot \ln x\ dx \qquad \int_{-2}^{5} (2x-1) \cdot e^{2x}\ dx \qquad \int_0^{\pi} x^2 \cdot \sin x\ dx$$

3. Determine the following definite integrals:

$$\int_0^{4} \frac{\sqrt{x}}{1+2\sqrt{x}}\ dx \qquad \int_{0,6}^{0,8} \frac{1}{x \cdot \sqrt{1-x^2}}\ dx \qquad \int_0^{2} \frac{e^x-1}{e^x+1}\ dx$$

11. Lesson 11

11.1. Improper Integral

In this section the concept of the definite integral will be extended to nonbounded intervals and to nonbounded functions.

11.1. Definition Let $-\infty \leq a < b \leq +\infty$ and I be the interval whose left-hand endpoint is a and the right-hand endpoint is b. (I can be any type of intervals.) Let $f : I \to \mathbb{R}$ be a function and suppose that f is integrable over every closed and bounded subinterval of I (this is the case for example if f is continuous). We say that the improper integral

$$\int_a^b f(x)\ dx$$

is convergent if for some $c \in (a, b)$ the following limits exist and are finite:

$$\lim_{t\to a+} \int_t^c f(x)\ dx\,, \qquad \lim_{s\to b-} \int_c^s f(x)\ dx\,.$$

In this case the value of the improper integral is

$$\int_a^b f(x)\ dx := \lim_{t\to a+} \int_t^c f(x)\ dx + \lim_{s\to b-} \int_c^s f(x)\ dx\,.$$

It can be proved that the convergence and the value of the improper integral are independent of c.

Some special cases of the improper integral follows.

<u>Case 1.</u> Let $I = [a, b]$ a closed and bounded interval and $f \in R[a, b]$. Let $c \in (a, b)$ and denote by F the integral function of f vanishing at c. Then the improper integral can be written as:

$$\int_a^b f(x)\ dx := \lim_{t\to a+} \int_t^c f(x)\ dx + \lim_{s\to b-} \int_c^s f(x)\ dx = \lim_{t\to a+} (-F(t)) + \lim_{s\to b-} F(s)\,.$$

Since F is continuous at a and b, the above computation can be continued so:

$$\int_a^b f(x)\ dx = -F(a) + F(b) = -\int_c^a f(x)\ dx + \int_c^b f(x)\ dx =$$

$$= \int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx$$

in the original sense of the definite integral. So in this case the original and the improper integral are the same.

Case 2. Let $I = [a, b)$. Using similar arguments (the continuity of the integral function) it can be proved that the improper integral can be computed as follows:

$$\int_a^b f(x)\,dx := \lim_{s\to b-} \int_a^s f(x)\,dx\,.$$

Case 3. Let $I = (a, b]$. Then – also by the the continuity of the integral function – the improper integral can be computed as follows:

$$\int_a^b f(x)\,dx := \lim_{t\to a+} \int_t^b f(x)\,dx\,.$$

Almost all the properties of the definite integral can be extended to the improper integral. Here we will discuss only the Newton-Leibniz formula.

11.2. Theorem *Using the previous notations let $f : I \to \mathbb{R}$ be a function and suppose that f is integrable over every closed and bounded subinterval of I. Suppose that $F : I \to \mathbb{R}$ is continuous on I and $F'(x) = f(x)\ \ (x \in intI)$. Then*

1. $\lim\limits_{t\to a+} \int_t^c f(x)\,dx$ *is finite* $\Leftrightarrow \lim\limits_{t\to a+} F(t)$ *is finite;*

2. $\lim\limits_{s\to b-} \int_c^s f(x)\,dx$ *is finite* $\Leftrightarrow \lim\limits_{s\to b-} F(s)$ *is finite;*

3. *In this case the improper integral can be computed as follows:*

$$\int_a^b f(x)\,dx = \lim_{s\to b-} F(s) - \lim_{t\to a+} F(t)\,.$$

Proof. By the Newton-Leibniz formula we can write:

$$\lim_{t\to a+} \int_t^c f(x)\,dx = \lim_{t\to a+} (F(c) - F(t)) = F(c) - \lim_{t\to a+} F(t)\,.$$

From here the first statement follows. The second statement can be proved similarly. Finally:

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{t\to a+} \int_t^c f(x)\,dx + \lim_{s\to b-} \int_c^s f(x)\,dx = \lim_{t\to a+} (F(c) - F(t)) + \\ &\quad + \lim_{s\to b-} (F(s) - F(c)) = F(c) - \lim_{t\to a+} F(t) + \lim_{s\to b-} F(s) - F(c) = \\ &= \lim_{s\to b-} F(s) - \lim_{t\to a+} F(t)\,.\end{aligned}$$

□

11.3. Remark. Let us denote the difference $\lim_{s\to b-} F(s) - \lim_{t\to a+} F(t)$ by $[F(x)]_a^b$. Using this notation the Newton-Leibniz Formula for the improper integral has the same form as the original one. Furthermore if $a, b \in I$ then – because of the continuity of F:

$$\lim_{s\to b-} F(s) = F(b) \quad \text{and} \lim_{t\to a+} F(t) = F(a)\,,$$

so we can realize that the Newton-Leibniz formula for the improper integral contains the common Newton-Leibniz formula as a special case.

11.2. Homeworks

1. Determine the following improper integrals:

$$\int_{-\infty}^{0} \frac{1}{(2x-1)^2}\,dx \qquad \int_{3}^{+\infty} \frac{1}{\sqrt{(x-1)^3}}\,dx \qquad \int_{-\infty}^{+\infty} \frac{1}{2+3x^2}\,dx \qquad \int_{0}^{+\infty} (x-1)\cdot e^{-x}\,dx$$

2. Determine the following improper integrals:

$$\int_{0}^{\pi/2} \frac{1}{\cos^2 x}\,dx \qquad \int_{0}^{2} \frac{1}{\sqrt{4-x^2}}\,dx$$

12. Lesson 12

12.1. Applications of the definite integral

In this section $[a,b] \subset \mathbb{R}$ is a closed bounded interval.

12.1. Theorem *[the area of an x-normal region] Let $f, g : [a,b] \to \mathbb{R}$ be continuous functions and suppose that*

$$\forall\, x \in [a,b] : \quad f(x) \le g(x)\,.$$

Then the area of the region

$$\big\{(x,y) \in \mathbb{R}^2 \;\mid\; a \le x \le b,\; f(x) \le y \le g(x)\big\} \subset \mathbb{R}^2$$

is equal to

$$\int_a^b (f(x) - g(x))\; dx\,.$$

Remark that the region in the theorem is called an x-normal region in $\mathbb{R}^2$.

12.2. Theorem *[arc length of a function graph] Let $I \subset \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$, $f \in D$, $f' \in C$. If $[a,b] \subset I$ then the arc length of the curve $\{(x,\, f(x)) \mid a \le x \le b\}$ is*

$$\int_a^b \sqrt{1 + (f'(x))^2}\; dx\,.$$

12.3. Theorem *[volume of a solid of revolution] Let $f : [a,b] \to \mathbb{R}$ be continuous function and suppose that $f(x) \ge 0$ $(x \in [a,b])$. Revolve its graph about the x-axis. Then the volume of the solid of revolution*

$$\{(x,y,z) \in \mathbb{R}^3 \;\mid\; a \le x \le b,\; y^2 + z^2 \le (f(x))^2\}$$

is equal to

$$\pi \cdot \int_a^b (f(x))^2\; dx\,.$$

12.4. Theorem *[area of the surface of revolution] Let $I \subset \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$, $f \in D$, $f' \in C$. Suppose that $f(x) \geq 0$ $(x \in [a, b])$. If $[a, b] \subset I$ then the area of the surface of revolution*

$$\{(x, y, z) \in \mathbb{R}^3 \mid a \leq x \leq b,\ y^2 + z^2 = (f(x))^2\}$$

is equal to

$$2\pi \cdot \int_a^b f(x) \cdot \sqrt{1 + (f'(x))^2}\ dx\,.$$

12.2. Homeworks

1. Determine the arc length of the following function graphs over the given interval.

$$y = x^{3/2}, \quad [0,\ 4]; \qquad y = \frac{x^2}{2}, \quad [0,\ 1]; \qquad y = \ln x, \quad [\sqrt{3},\ \sqrt{8}]$$

2. Find the areas of the regions enclosed by the given curves:

$$y = x^2 + 2x \text{ and } y = 4 - x^2; \qquad y = \frac{1}{x} \text{ and } y = 2,5 - x;$$

$$y = \sin x \text{ and } y = \frac{2}{\pi} \cdot x; \qquad y = x^4 \text{ and } y = 3x^2 - 2.$$

3. The following regions between the given curve and the x-axis are revolved about the x-axis to generate a solid. Find their volumes.

$$y = \frac{1}{x}, \quad x \in [1,\ 3]; \qquad y = x \cdot e^x, \quad x \in [0,\ 1];$$

$$\sqrt{x} + \sqrt{y} = 1, \quad x \in [1,\ 4]; \qquad y = \frac{x^3}{3}, \quad x \in [1,\ 2].$$